Issue No.12 PRACTICAL MATHEMATICS Volume No.2

250 LECTURES ON MATHEMATICS · PUBLISHED SERIALLY · THREE TIMES EACH MONTH

ISSUE No.12

# PRACTICAL MATHEMATICS

THEORY AND PRACTICE WITH MILITARY AND INDUSTRIAL APPLICATIONS

**APPLIED MATHEMATICS**

**Electricity**

*Ohm's and Kirchhoff's Laws*
*Condensers*
*Alternating Current*

**Gunnery and Ballistics**

*Preparation of Fire*
*The Trajectory*
*Dispersion and Probability*

—ALSO—

*Mathematical Tables and Formulas*
*Self-Tests and Problems*

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EDITOR: REGINALD STEVENS KIMBALL ED.D.

ISSUE 12 **Practical Mathematics** VOLUME 2

REGINALD STEVENS KIMBALL, Editor



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*PRACTICAL MATHEMATICS is published three times a month by the National Educational Alliance, Inc., Office of publication at Washington and South Avenues, Dunellen, N. J. Executive and editorial offices, 37 W. 47th St., New York, N. Y. John J. Crawley, President; A. R. Mahony, Vice President and Business Manager; Frank P. Crawley, Treasurer. Issue No. 12, July 30, 1943. Entered as second-class matter April 10, 1943, at the Post Office at Dunellen, N. J., under the Act of March 3, 1879. Printed in the U.S.A. Price in the U.S.A. 35c a copy; annual subscription at the rate of 35c a copy.* 

## CHATS WITH THE EDITOR

PROBABLY nothing in warfare seems more "practical" than the firing of a gun. While we know that there is much to the science of modern warfare in addition to the actual "shooting", we are somehow prone to distinguish the serious business of waging battle from the sport of playing games through a count of the combatants killed or wounded and ships or planes lost or crippled as a result of gunfire.

As youngsters, we played at warfare—battling imaginary Indians and making many a redskin bite the dust. With our wooden pistols, all that we had to do was close our eyes and pull the trigger and—lo! the job was accomplished. Not much mathematics in that!

If you have ever visited a shooting gallery, you undoubtedly remember how much easier it was to hit the stationary targets than it was to topple over the moving ones. After a certain amount of practice, having gotten the "range", we manage to do fairly well, however. When you multiply the difficulties by increasing the range and having to figure on both target and gun moving, or when you find it necessary to aim at an invisible target, you begin to appreciate that mathematics may have a part in the winning of the war. Dr. Littauer's article will help you to realize the utilization of mathematical principles in solving these problems.

The electrical engineer seems to live with a slide rule in his hand. After reading Dr. Hefner's article on the mathematics of electricity, you will be better able to appreciate why this is so. As in our other articles on the application of mathematics, we are making no pretension of giving an exhaustive treatise on the subject of electricity, but are merely seeking to show how mathematics is brought into play in working out the problems involved.

Although mathematics is an exact science, the degree of exactness depends upon the formulas and instruments used and upon the user's adeptness or expertness. One of my colleagues who makes no pretense of being a mathematician came into my office the other day just in time to hear two members of my staff discussing an "approximately correct" answer. Puzzledly, he exclaimed, "Why, I always thought that, in mathematics, either a thing was all right or it was all wrong!"

Perhaps you have encountered people who have had the same attitude toward mathematics. Maybe you have at some time been numbered among them.

Preciseness, definiteness, accuracy —much depends upon the point of view. If one is down to his last dollar, every cent will be counted over and expended with extreme care; if one's capital is expressed in millions, we are less concerned with knowing what follows the decimal point. In measuring a distance involving miles, it would be ridiculous to demand that we compute accuracy down to ten-thousandths of an inch; on the other hand, it would be folly

to try to measure the diameter of a small bore with a yardstick.

Given a choice of formulas or methods to be used in achieving a desired result, you find that the secret of success lies in knowing what formula to use to attain the desired degree of accuracy. If you go through an involved bit of computation to achieve a result which is unnecessarily fine, you are putting yourself to a burdensome bit of work which might have been avoided. On the other hand, you deprive your final answer of the claim to accuracy which it should have if you use a clumsy instrument or formula when a precision instrument or a more exact formula should have been employed.

In your own work, you probably realize the degree of definiteness which is desirable in any instance. By the time you have completed this course, we hope that you will understand equally well the distinctions which should be made in each of the fields covered.

There is something more than selecting the right formula, however. Once the formula has been applied and the proper substitutions made, your work descends to "straight arithmetic" or simple algebra. Unless you have trained yourself to accuracy in your arithmetical computations, no formula, however precise, will be safe in your hands. Until you are sure that you can achieve one hundred per cent correctness in the fundamental operations, then, you must school yourself to more practice in arithmetic.

Unless you know that dividing by a fraction gives you a result greater than the figure with which you began, unless you can discover the approximate correctness of your figuring from a simple comparison (since $2\times3=6$ and $3\times3=9$, the answer to $2.147\times3.264$ should be slightly greater than 6, but less than 9), you are in a fair way to "mess up" your work just at the point where you think you have conquered it. Often a diagram will help you to see relationships existing among the quantities which you are computing. As in the two articles in this issue, diagrams may be brought into play to picture in small scope and at close range problems which are too large to be visualized in their entirety. Here, again, precision and accuracy are of prime importance, for an incorrect diagram may prove misleading.

After this first "roughing out" of a mathematics course through the study of the issues of PRACTICAL MATHEMATICS, you will probably decide that you want to embark on a program of study in your chosen branch of mathematics that may well extend over a period of years. As you continue to grow in your knowledge of what is involved in mathematical computations, you will instinctively come to recognize the degree of correctness which is essential in a given situation.

Whenever it is possible to utilize a table to replace lengthy calculations, or to employ a formula to assist you in arriving at a result more quickly, you should, by this time, have formed the habit of doing so. There are many little tricks of extension along this line that you can pick up from time to time. Having to multiply 27 by 48, if we think of 27 as $3^3$ and of 48 as $2^4\cdot 3$, we may combine these exponentially to achieve $2^4\cdot 3^4$, which is $6^4$ or $36^2$. Then, by reference to a table of squares, we get 1296. Thus, by breaking the original numbers up into their factors, we have arrived at a means of performing practically all of the work mentally. (If you prefer the slide rule for this purpose, you can, of course, eliminate practically all of the mental effort.)

As the war continues, we find more and more people seeking a further knowledge of mathematics. One of my close friends, who has confined his

thinking to historical subjects, now finds himself faced with the necessity of mastering meteorology. Another, who has devoted his life to teaching foreign languages, is being required by his university to take courses in the teaching of mathematics, in the expectation that changing enrollments within the university will lead to a decrease in registrations in the foreign-language field and to such an increase in registrations in mathematics that teachers will have to be "drafted" from other departments. These experiences might be duplicated over and over again. The demand for mathematical knowledge, applied to the war effort, keeps pace with the progress of the war.

With the coming of the peace, this demand is likely to continue, for there will probably be a new surge of construction and a new demand for the many devices which people have been forced to forego during the "emergency". Industry will forge ahead with new developments, all of which will entail a knowledge of mathematics and the ability to make applications to the new problems which will be constantly arising.

Already our manufacturers are giving us glimpses, in their advertising, of new and improved models of refrigerators, radios, helicopters which will be available to all of us at reduced prices just as soon as the priorities of the war-time exigencies can be lifted. Unable to make such purchases now, many people are saving up for the future, against the day when restrictions will be removed and trade may once again be permitted. The researches already undertaken along these lines will, if we may judge from past experience, entail still further researches and more opportunities to utilize the mathematical formula for predictive and exploratory excursions into the industrial field.

We may envision, then, a great demand in the post-war period for men and women who have sufficient knowledge of mathematics to be able to assist, in the shop and in the laboratory, in carrying forward the experiments which will be demanded. The readers of PRACTICAL MATHEMATICS who have given attention to the applications suggested in these issues will find that they have a good start toward such peace-time use of mathematics.

In all of the applied issues of PRACTICAL MATHEMATICS, my colleagues have developed the subjects in such a way that the reader will find the articles of value in their peace-time pursuits as well as at the present moment.

After the First World War, many a young man who got his first glimpse of radio through his service in the Navy found that he had thereby acquired the foundation for a life-time career in private business. It is just as probable that many who are now "taking up" various subjects because of the part that they may thereby hope to play in winning the war may, likewise, find that they have equipped themselves with a new string to their bow for the post-war period.

Most of us are looking forward to the period after the war with some questioning as to the part we shall be called upon to play, as individuals and as a nation. While no one can say with certainty just what kind of world the post-war world will be, we may be sure that the person who has equipped himself with the ability to use the tools of civilization will be in a somewhat better position to take advantage of new developments and to be of greater service to mankind.

Because of the nature of the demands which will be forthcoming, we may expect that a knowledge of mathematics will prove a great asset in almost every field of endeavor. Let us resolve not to get "caught short" again. The war found us, as a nation, mathematically unprepared.

There is yet time, before the peace, to make good this deficit and to store up a reserve of mathematical ability, since it increases rather than shrinks with use, for the perplexing days of the adjustment period.

Issue Number Thirteen will be devoted to navigation and radio. In the first of these articles, Dr. Michael will link problems in navigation with the work in trigonometry which has preceded and will show the part that navigation plays both on the sea and in the air. In the second article, Mr. Maedel will show how the science of radio depends upon mathematics for its development.

R.S.K.

## ABOUT OUR AUTHORS

RALPH A. HEFNER, author of the article on the mathematics of electricity, has been faculty member of the Georgia School of Technology since 1929, when he began as an Instructor in Mathematics. He attained the rank of Professor in 1937 and is serving in that capacity today.

It may be of passing interest to the readers of PRACTICAL MATHEMATICS to know that Dr. Hefner is also active in the research field of aerodynamics. He is the co-author of two important papers on that subject, both dealing with the lifting air screw or helicopter rotor.

Dr. Hefner was born in Blue Field, West Virginia, in 1902. He earned the degree of Bachelor of Science at Roanoke College in 1925 and taught physics there for a short period. He then entered the University of Chicago for post-graduate work, obtaining the degree of Master of Science in 1927 and that of Doctor of Philosophy in 1931. He is a member of the Georgia Academy of Science, the Mathematical Association of America, and the American Mathematical Society.

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ONE of the more difficult assignments of the editors of this publication was to select a man well qualified to prepare the section on gunnery. We feel that the choice of Sebastian B. Littauer, who from 1940 to 1942 served as Assistant Professor of Mathematics at the United States Naval Academy at Annapolis, provides our readers with a manuscript which justifies our efforts.

Mr. Littauer was born in New York City in 1900. After receiving his preliminary education in the public schools of that city, he entered Rensselaer Polytechnic Institute, from which he obtained the degree of Chemical Engineer in 1920. He did graduate work at New York University in 1924 and 1925, and then entered Columbia University, where he was awarded the degree of Master of Arts in 1928. There followed additional work at Massachusetts Institute of Technology, which won for him the degree of Doctor of Science in 1930 and also earned for him a National Research Council Fellowship in Mathematics with tenure at Harvard.

From 1922 to 1940, Mr. Littauer was engaged variously at teaching posts in the New York City public high schools, Hunter College, M.I.T., Harvard, and the United States Naval Academy, where he was made an Assistant Professor in 1940. He has also served as a consultant for the Weems System of Navigation and as senior industrial specialist for the War Production Board.

Mr. Littauer has done much work in mathematical research and has published a great number of papers in that connection.

## · ELECTRICITY ·

By Ralph A. Hefner, Ph. D.

THE first requirement for a successful solution of any problem in electricity is a clear understanding of the various units of measurement used in electricity. For that reason, it is desirable to begin with a list of the fundamental units employed in the measurement of electrical quantities.

### UNITS OF MEASUREMENT

The *coulomb* is the unit quantity of electricity. One coulomb is approximately equal to the charge of 6,240,000,000,000,000,000 electrons. Written more conveniently, the number is indicated as $6.24 \times 10^{18}$*.

The *ampere* is the unit of electric current. If one coulomb flows through a wire per second, the current is that of one ampere.

The force which causes electricity to flow through a conductor is called *electromotive force*, and is usually abbreviated to e.m.f. The unit e.m.f. is one *volt*. It is that force or potential which will cause a current of one ampere to flow through a resistance of one *ohm*.

Every conductor resists the flow of current to some extent. The unit of resistance is the *ohm*. It is the resistance at 0° C. of a column of mercury 106.300 cm. in length having a mass of 14.4521 grams.

The *watt* is the unit of electrical power. For direct current circuits, the power in watts is the product of the voltage times the current—that is,

$$P = EI.$$

If a wire carries a current of 6 amperes under a pressure of 10 volts, the power delivered is $10 \times 6 = 60$ watts.

The joule is a unit of electrical energy. One joule is equal to one *watt-second*; that is, one watt is a rate of one joule per second. Since the watt is a rate of doing work, it is comparable to horse power. Thus, one joule is equal to 0.738 foot-pound.

A current of one ampere under a pressure of one volt produces one joule or 0.738 foot-pound of energy per second. Six amperes under 100 volts

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* The latter method of writing the number is one commonly employed in electricity and is a method with which the student should be thoroughly familiar.

pressure produce $6 \times 100 = 600$ joules, or $0.738 \times 600 = 442.8$ foot-pounds per second.

A *kilowatt* = 1,000 watts. It is equal to 1,000 joules per second, or 738 foot-pounds per second.

One horse power = 550 foot-pounds per second; so

$$\text{one kilowatt} = \frac{738}{550} = 1.341 \text{ horse power.}$$

$$\text{One horse power} = \frac{1}{1.341} \text{ kilowatts} = 0.746 \text{ kilowatts.}$$

The *kilowatt-hour* is a unit of electrical energy larger than the joule. It is commonly used in commercial light and power circuits. The kilowatt-hour is the work done in one hour at a rate of one kilowatt.

$$\begin{aligned} \text{One kilowatt} &= 1{,}000 \text{ joules per second} \\ &= 3{,}600{,}000 \text{ joules per hour.} \end{aligned}$$

Thus, $\qquad \text{one kilowatt-hour} = 3{,}600{,}000 \text{ joules.}$

The *calorie* is a unit of heat. It is the amount of heat required to raise the temperature of one gram of water 1° C. The calorie is also connected with energy through the relation,

$$\text{one calorie} = 4.18 \text{ joules.}$$

As one kilowatt-hour equals 3,600,000 joules, one kilowatt-hour equals $3{,}600{,}000 \div 4.18 = 860{,}000$ calories.

To get an idea of this amount of heat, consider the problem of how many gallons of water it would bring to boil.

One gallon of water weighs approximately 8.3 lb. or 3765 grams. To raise the temperature of one gram of water at 20° C. to 100° C. requires $100 - 20 = 80$ calories.

One gallon requires $3765 \times 80 = 301{,}200$ calories. Hence, the kilowatt-hour supplies sufficient heat to bring $\frac{860{,}000}{301{,}200} = 2.85$ gallons, or 11.4 quarts of water to boiling temperature.

## Computing thermal capacity

The *thermal capacity* of any object is the number of calories required to raise its temperature 1° C.

One quart of water weighs approximately 941 grams. Hence, the thermal capacity of one quart of water is 941 calories. The thermal capacity of a quart kettle full of water is higher than 941 calories because the metal has to be heated as well as the water.

### *Illustrative Problem*

A metal pot containing 3 pints of water has a thermal capacity of 1,500. If the pot is placed over a 750-watt electric heater, how long will it take to bring it to boiling temperature if the original temperature is 12° C.?

To raise the pot from 12° C. to 100° C. requires

$$88 \times 1{,}500 = 132{,}000 \text{ calories.}$$

The heater supplies 750 watts

$$= 750 \text{ joules per second}$$

$$= \frac{750}{4.18} \text{ calories per second.}$$

The time taken to boil is

$$\frac{132{,}000}{\frac{750}{4.18}} \text{ seconds} = 735.68 \text{ seconds} = 12\tfrac{1}{4} \text{ minutes.}$$

The *henry* is the unit of inductance. A circuit has a self-inductance of one henry when a counter e.m.f. of one volt is generated by a rate of change of current of one ampere per second.

The *farad* is the unit of capacitance. It is one coulomb per volt. Thus, a condenser of one farad capacitance is one on which one volt will put one coulomb of positive electricity on one plate and one coulomb of negative electricity on the other.

### *Measuring frequency*

A current that is reversed at regular intervals is called an *alternating current.* When such a current changes from zero to maximum, to zero, to maximum in the opposite direction, and finally returns to zero, it has completed what is called one *cycle*. The number of times that such a cycle is completed in one second is the *frequency* of the alternating current. The current supplied in most cities in the United States is 60-cycle alternating current.

### *Utilizing the units*

The units defined above are not always convenient to use in electrical problems. In many instances, it would be impossible to adopt any one unit that would always be of the right size. This is due to the fact that the units as defined in the fields of radio and electrical engineering include extremely wide ranges in values. In any given problem, one might be dealing with values ranging from one-millionth of a unit to several million times the same unit.

## ABBREVIATIONS

To prevent the necessity of writing long strings of zeros either before or after a number, a system of units based on powers of ten has been developed. Thus, for each electrical quantity, there is a wide choice of units in which it can be measured. The more commonly used of these units are defined below:

The *milli-unit* is one-thousandth of a unit. Thus,

2 watts = 2,000 milliwatts; 250 milliamperes = 0.25 amperes.

It is abbreviated m. Thus,

17.6 mh = 17.6 millihenrys = 0.0176 henrys.

The *micro-unit* is one-millionth of a unit. Thus,

1 volt = 1,000,000 microvolts; 3.5 microfarads = 0.0000035 farads.

The micro-unit is designated by the Greek letter, $\mu$. Thus,

$$15\ \mu f = 15 \text{ microfarads.}$$

The *micro-micro-unit* is one-millionth of one-millionth of a unit. It is designated by $\mu\mu$ and is seldom used except for farads. Thus,

$$750\ \mu\mu f = 0.00075\ \mu f = 0.00000000075\ f = 7.5 \cdot 10^{-10}\ f.$$

The *kilo-unit* is one thousand units. Thus,

$$12 \text{ kilowatts} = 12{,}000 \text{ watts.}$$

It is abbreviated k. Hence,

$$780 \text{ kc} = 780 \text{ kilocycles} = 780{,}000 \text{ cycles.}$$

The *meg-unit* is one million units. Thus,

$$12 \text{ megohms} = 12{,}000{,}000 \text{ ohms}; 7{,}600{,}000 \text{ cycles} = 7.6 \text{ megacycles.}$$

## CONVERSION FACTORS

The following table gives the factor whereby any one of the multi-units can be converted into units or vice versa:

| Multiply Number of | By | To Obtain Number of |
|---|---|---|
| units | $10^{-3}$ | kilo-units |
| units | $10^{-6}$ | meg-units |
| units | $10^{3}$ | milli-units |
| units | $10^{6}$ | micro-units |
| units | $10^{12}$ | micro-micro-units |
| kilo-units | $10^{3}$ | units |
| meg-units | $10^{6}$ | units |
| milli-units | $10^{-3}$ | units |
| micro-units | $10^{-6}$ | units |
| micro-micro-units | $10^{-12}$ | units |

### TEST YOUR KNOWLEDGE OF ELECTRICAL UNITS

1 How many foot-pounds of energy are produced by an electric current of 12 amperes at an e.m.f. of 1 volt in 10 minutes?

2 The current through a radio amplifier is $3.6 \cdot 10^{-5}$ amperes. How many microamperes is this?

3 An electric heating unit which delivers 600 watts is placed in 2 kg. of stirred water. How rapidly will the temperature of the water rise in C. degrees per minute? (Neglect heat losses.)

4 How long will it take a 2-kw heating unit to boil off 2 pounds of water initially at 60° F.? (Heat of vaporization of water: 540 calories per gram.)

5 A photophone light consumes at 12 volts direct current of 0.67 amperes. How many watts does it consume?

6 A coil has an inductance of 0.0000067 henrys. How many microhenrys does this equal?

7 How many electrons would compose one micro-coulomb?

## OHM'S LAW AND RESISTANCE

Ohm's law states that the current flowing through a conductor is proportional to the difference in potential between the two ends of the conductor. Thus, if the difference in potential between the ends of a particular conductor be divided by the amount of current flowing, the quotient will be a constant, no matter how the potential difference is changed. This constant is called the *resistance* of the conductor.

Stated more compactly, Ohm's law reads:

$$\frac{\text{potential difference}}{\text{current}} = \text{resistance.}$$

If $E$ = the potential difference in volts
$I$ = the current in amperes
$R$ = the resistance in ohms,
then Ohm's law becomes

$$R=\frac{E}{I}. \qquad \text{I}$$

Ohm's law as given in equation I is in convenient form to find $R$ when $E$ and $I$ are known. Multiplying both sides of I by $I$ gives

$$E=IR, \qquad \text{II}$$

a more convenient form to use when $R$ and $I$ are known.

Dividing both sides of II by $R$ yields

$$I=\frac{E}{R}, \qquad \text{III}$$

the form to use when $E$ and $R$ are known.

Fig. 1 represents a circuit that will clearly illustrate the various forms of Ohm's law. The direct current generator, $G$, has a load across its output in the form of the resistance, $R$. $V$ is a voltmeter connected across the resistance. $A$ is an ammeter placed in the circuit to read the amount of current, $I$, flowing around the circuit. If the electricity consumed by the two meters is neglected, the voltmeter registers the electromotive force, $E$, being generated.

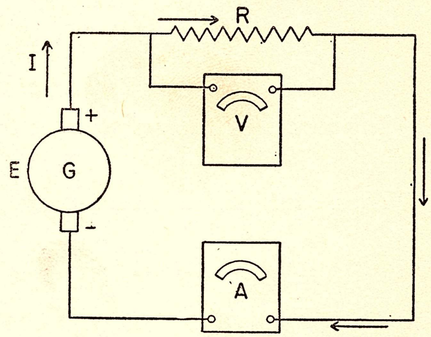


Fig. 1

If the voltmeter reads 115 v while the ammeter reads 0.5 a, then $E=115$, $I=0.5$, and equation I gives the resistance, $R$, as

$$R=\frac{115}{0.5}=230\ \Omega=230 \text{ ohms.}$$

If the resistance is changed to 200 Ω and the ammeter reads 0.6 a, then $R=200$, $I=0.6$, and equation II gives the e.m.f. as

$$E=0.6\times 200=120 \text{ volts.}$$

If the voltmeter reads 110 v and the resistance is known to be 250 Ω, then equation III says that the ammeter would read

$$I=\frac{110}{250}=0.44 \text{ amperes.}$$

### Resistors

Resistors may be connected either in series or in parallel.

#### RESISTORS IN SERIES

When two or more resistors are connected as shown in Fig. 2, they are said to be connected in series. The total resistance of any number of resistors so connected is found by adding together their separate

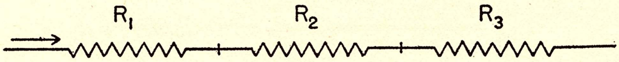



Fig. 2

resistances; thus, $R=R_1+R_2+R_3+\text{etc.}$ **IV**

If in Fig. 2, $R_1=1.5\ \Omega$, $R_2=3.6\ \Omega$, and $R_3=8.7\ \Omega$, then the total resistance would be $R=1.5+3.6+8.7=13.8$ ohms.

#### RESISTORS IN PARALLEL

If two or more resistors are connected as shown in Fig. 3, they are said to be connected in parallel. When resistors are connected in parallel, their combined resistance is always less than the resistance of any of the separate resistors. If $R_1$, $R_2$, $R_3$, etc. are the resistances of a number of resistors connected in parallel and if $R$ is their combined resistance, then

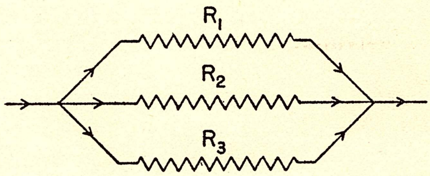



Fig. 3

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\text{ etc.} \qquad \text{V}$$

*Illustrative Example A*

Suppose that the resistors in Fig. 3 have the resistances,

$$R_1=4\ \Omega,\ R_2=5\ \Omega,\ R_3=6\ \Omega.$$

Their combined resistance would then be given by

$$\frac{1}{R}=\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{15+12+10}{60}=\frac{37}{60}$$

$$R=\frac{60}{37}\text{ ohms.}$$

The method of computing $R$ used in the preceding example is not recommended unless the common denominator is a simple one. When it is not, the method given below should be followed:

*Illustrative Example B*

Five resistances, $R_1=1.15\ \Omega$, $R_2=2.36\ \Omega$, $R_3=0.96\ \Omega$, $R_4=1.07\ \Omega$, and $R_5=4.27\ \Omega$, are connected in parallel. What is their combined resistance?

Equation V states

$$\frac{1}{R}=\frac{1}{1.15}+\frac{1}{2.36}+\frac{1}{0.96}+\frac{1}{1.07}+\frac{1}{4.27}.$$

Select 4.27, the largest denominator, and multiply the equation through by 4.27 $R$; thus,

$$4.27=\frac{4.27}{1.15}R+\frac{4.27}{2.36}R+\frac{4.27}{0.96}R+\frac{4.27}{1.07}R+R.$$

Reduce each fraction to a decimal by division.

$$4.27=3.71R+1.81R+4.45R+3.99R+R$$
$$4.27=(3.71+1.81+4.45+3.99+1)R$$
$$4.27=14.96R$$
$$R=\frac{4.27}{14.96}=0.2855\ \Omega.$$

## *Applications of Ohm's law*

In the development of series circuits, parallel circuits, and series-parallel circuits, we find it necessary to apply Ohm's law in different ways.

### APPLICATIONS TO A SERIES CIRCUIT

Three resistors, having resistances, $R_1=80\ \Omega$, $R_2=180\ \Omega$, and $R_3=700\ \Omega$, respectively, are connected in series to a 120-v direct-current generator (Fig. 4). How much current flows around the circuit and what is the drop in voltage across each resistor?

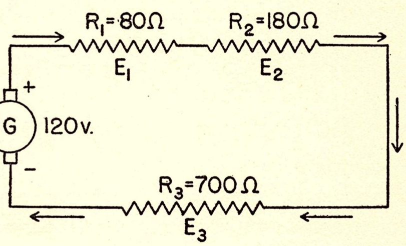


Fig. 4

The total resistance of the circuit is given by equation IV:

$$R=80+180+700=960 \text{ ohms.}$$

By formula III, the amount of current flowing is

$$I=\frac{E}{R}=\frac{120}{960}=0.125 \text{ amperes.}$$

To find the voltage drop across each resistor, use formula II; thus:

$$E_1=IR_1=0.125\times\ 80=10 \text{ volts}$$
$$E_2=IR_2=0.125\times 180=22.5 \text{ volts}$$
$$E_3=IR_3=0.125\times 700=87.5 \text{ volts.}$$

The results can be checked by noting that the sum of all the voltage drops around the circuit is equal to the total e.m.f. produced by the generator; thus:

$$E=E_1\ +E_2\ \ +E_3$$
$$120 \text{ v}=10 \text{ v}+22.5 \text{ v}+87.5 \text{ v.}$$

### APPLICATIONS TO A PARALLEL CIRCUIT

Three resistors, with resistances, $R_1=200\ \Omega$, $R_2=150\ \Omega$, and $R_3=300\ \Omega$, are connected in parallel across a 120-v direct-current gen-

erator (Fig. 5). What is the total current supplied by the generator and how much current is flowing through each resistor?

The combined resistance of the circuit is found by means of formula V; thus:

$$\frac{1}{R}=\frac{1}{200}+\frac{1}{150}+\frac{1}{300}=\frac{3+4+2}{600}=\frac{9}{600}$$

$$R=\frac{600}{9}=\frac{200}{3}\text{ ohms.}$$

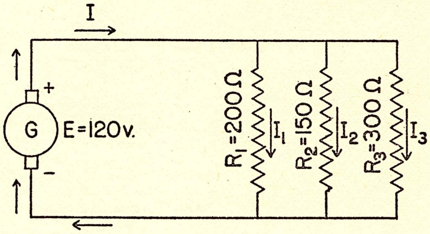



Fig. 5

By formula III, the total amount of current flowing is

$$I=\frac{E}{R}=\frac{120}{\frac{200}{3}}=\frac{360}{200}=1.8\text{ amperes.}$$

Formula III is also used to find the amount of current through each resistor; thus:

$$I_1=\frac{E}{R_1}=\frac{120}{200}=0.6\text{ a}$$

$$I_2=\frac{E}{R_2}=\frac{120}{150}=0.8\text{ a}$$

$$I_3=\frac{E}{R_3}=\frac{120}{300}=0.4\text{ a.}$$

Again the results can be checked, as the sum of the currents through the three resistors should be equal to the total current. That is,

$$I=I_1 \quad +I_2 \quad +I_3$$
$$1.8\text{ a}=0.6\text{ a}+0.8\text{ a}+0.4\text{ a.}$$

## APPLICATIONS TO A SERIES-PARALLEL CIRCUIT

Seven resistors are connected to a 120-volt D.C. generator in a series-parallel circuit, as shown in Fig. 6. Find (a) the total amount of current flowing, (b) the four voltage drops around the circuit, and (c) that portion of the total current which flows through each resistor connected in parallel with another.

The first step is to find the combined resistance of each group of resistors connected in parallel. If $R_{AB}$ is the combined resistance of the two resistors, $R_1$ and $R_2$, then, by equation V,

$$\frac{1}{R_{AB}}=\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{400}+\frac{1}{600}=\frac{3+2}{1200}=\frac{5}{1200}$$

$$R_{AB}=\frac{1200}{5}=240\ \Omega.$$

If $R_{CD}$ is the combined resistance of the three resistors, $R_4$, $R_5$, and $R_6$, then

$$\frac{1}{R_{CD}}=\frac{1}{R_4}+\frac{1}{R_5}+\frac{1}{R_6}=\frac{1}{200}+\frac{1}{400}+\frac{1}{1200}=\frac{6+3+1}{1200}=\frac{10}{1200}$$

$$R_{CD}=\frac{1200}{10}=120\ \Omega.$$

The total resistance of the circuit can now be computed. It is

$$R = R_{AB} + R_3 + R_{CD} + R_7$$

$$R = 240 + 100 + 120 + 140 = 600.$$

Since the total resistance, $R$, and the electromotive force, $E$, are known,

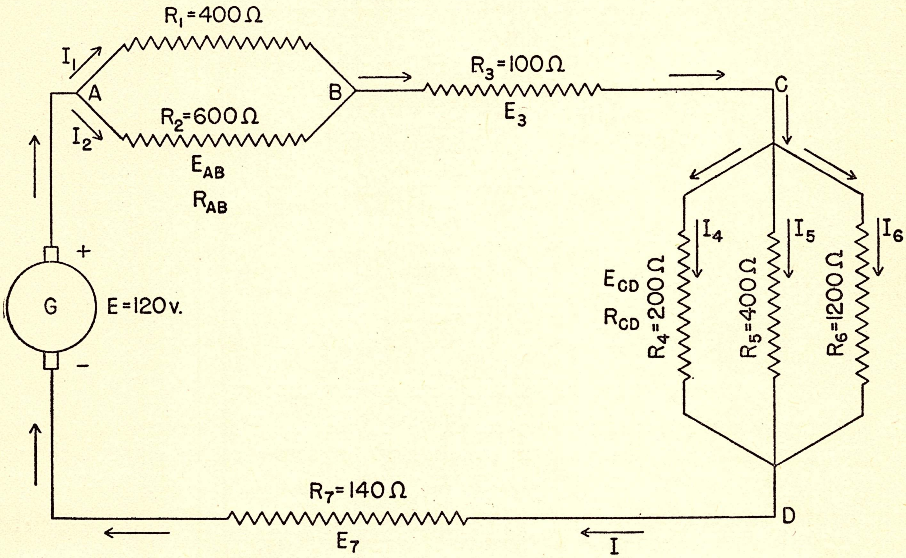



Fig. 6

equation III gives the total amount of current flowing; thus,

$$I = \frac{E}{R} = \frac{120}{600} = 0.2 \text{ amperes.}$$

Equation II is used to compute the various voltage drops around the circuit. If $E_{AB}$ is the drop from $A$ to $B$, then

$$E_{AB} = IR_{AB} = 0.2 \times 240 = 48 \text{ volts.}$$

The voltage drop across $R_3$ is

$$E_3 = IR_3 = 0.2 \times 100 = 20 \text{ volts.}$$

If $E_{CD}$ is the drop in voltage from $C$ to $D$, then

$$E_{CD} = IR_{CD} = 0.2 \times 120 = 24 \text{ volts.}$$

Finally, the drop across $R_7$ is

$$E_7 = IR_7 = 0.2 \times 140 = 28 \text{ volts.}$$

As the total e.m.f. should be equal to the sum of the voltage drops around the circuit, the results can be checked; thus,

$$E = E_{AB} + E_3 + E_{CD} + E_7$$

$$120 = 48 + 20 + 24 + 28.$$

To find that portion of the current which flows through each of the resistors connected in parallel, equation III is used. For the resistors, $R_1$ and $R_2$:

$$I_1=\frac{E_{AB}}{R_1}=\frac{48}{400}=0.12 \text{ a}$$
$$I_2=\frac{E_{AB}}{R_2}=\frac{48}{600}=0.08 \text{ a.}$$

As the total current flowing at $A$ must be equal to the sum of the currents through the two resistors, the above results can be checked as follows:

$$I=I_1 \quad +I_2$$
$$0.2 \text{ a}=0.12 \text{ a}+0.08 \text{ a.}$$

For the resistors, $R_4$, $R_5$, and $R_6$:

$$I_4=\frac{E_{CD}}{R_4}=\frac{24}{200}=0.12 \text{ a}$$
$$I_5=\frac{E_{CD}}{R_5}=\frac{24}{400}=0.06 \text{ a}$$
$$I_6=\frac{E_{CD}}{R_6}=\frac{24}{1200}=0.02 \text{ a.}$$

As a check on the results:

$$I=I_4 \quad +I_5 \quad +I_6$$
$$0.2 \text{ a}=0.12 \text{ a}+0.06 \text{ a}+0.02 \text{ a.}$$

### TEST YOUR KNOWLEDGE OF RESISTANCE

**8** A flashlight cell delivers an e.m.f. of 1.6 volts. The resistance of the bulb is 7.8 ohms and of the connections, and the cell itself, 0.2 ohms. (a) What is the current through the light? (b) How many watts does it deliver?

**9** Two searchlights are placed in series on a 60-volt direct-current circuit. The resistance of the circuit, exclusive of the lights, is 10 ohms. Light A, if placed alone in the circuit, would draw 4 amperes; light B, 3 amperes. How many amperes do they draw in series?

**10** A fuse link has a resistance of 0.2 ohms. It is a strip of lead weighing 3 grams. If a current of 60 amperes is suddenly passed through it, how long will the link take to melt? (Assume initial temperature, 20° C.; melting point of lead, 328° C.; specific heat, 0.033; heat of fusion, 5.8 calories.)

**11** What single resistance would be equivalent to 24 ohms, 18 ohms, 15 ohms, and 12 ohms in parallel?

**12** A circuit with an e.m.f. of 12 volts carries a current of 2 amperes through a light which has a resistance of 4 ohms. If 3 more lights of the same resistance are connected in parallel with it and e.m.f. is unchanged, what total current will pass through the circuit?

**13** A wet battery has an e.m.f. of 4 volts and an internal resistance of 0.6 ohms. The connections to the circuit have 0.4 ohms. If 3 lamps with resistances of 20, 20, and 40 ohms are connected in parallel, and a heating coil of 8 ohms is placed in the circuit between the 3 lamps and one terminal of the battery, how many watts is the heating coil delivering?

## Shunts

A *shunt* is a wire or bar of low resistance connected in parallel with another conductor such as a galvanometer or other instrument. The purpose of the shunt is to carry the bulk of the current so that only

a small portion of it passes through the galvanometer. The question that most frequently arises is to determine what resistance the shunt must have so that a suitable current will pass through the instrument.

*Illustrative Example*

A shunt is to be connected across a galvanometer the resistance of which is known to be 10 ohms. (See Fig. 7a.) What must be the resistance of the shunt if a current of only 0.02 amperes, out of a total current of 4 amperes, is to be allowed to pass through the galvanometer?

Fig. 7b is an equivalent-circuit diagram of the problem. $R_G$ is the resistance of the galvanometer, and $R_S$ is the resistance of the shunt.

As $R_G$ carries only 0.02 a, $R_S$ must carry 4 a$-$0.02 a$=$3.98 a. The drop in potential across both resistances is the same, and since $R_G$ and $I_G$ are both known, the drop is computed by equation II,

$$E=I_G R_G=0.02\times 10=0.2 \text{ volts.}$$

Using this result and equation I, we find the resistance of the shunt to be

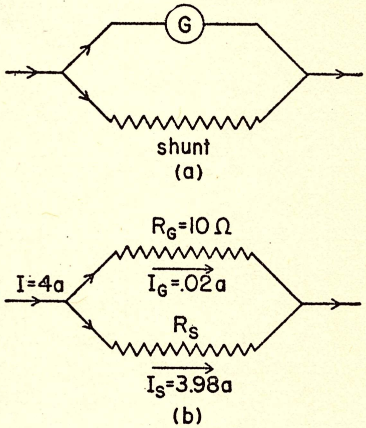


Fig. 7

$$R_S=\frac{E}{I_S}=\frac{0.2}{3.98}=0.0499=0.05 \text{ ohms, approximately.}$$

The problem can be generalized so as to let the galvanometer carry $\frac{1}{n}$ of the whole current. In such a case,

$$I_G=\frac{I}{n},\ I_S=I-\frac{I}{n}=\frac{(n-1)I}{n}.$$

The drop in potential is

$$E=I_G R_G=\frac{IR_G}{n}, \qquad \text{VI}$$

where $R_G$ is the resistance of the galvanometer.

The resistance in the shunt is, by equation I,

$$R_S=\frac{E}{I_S}=\frac{E}{\frac{(n-1)I}{n}}=\frac{nE}{(n-1)I}$$

Substituting the value of $E$ from equation VI gives

$$R_S=\frac{nE}{(n-1)I}=\frac{n}{(n-1)I}\times\frac{IR_G}{n}$$

$$R_S=\frac{R_G}{n-1}. \qquad \text{VII}$$

Expressed in words, formula VII states that the resistance of a shunt designed to carry all but $\frac{1}{n}$ of the current must be $\frac{1}{n-1}$ times the resistance of the galvanometer.

To apply formula VII to the example given above, let $R_G=10$ ohms. The expression, $\frac{1}{n}=\frac{0.02}{4}=\frac{1}{200}\cdot n$, therefore, has the value, 200. Hence,

$$R_S=\frac{10}{200-1}=\frac{10}{199}=0.05 \text{ ohms, approximately.}$$

A *tenth shunt* is a shunt intended to take nine-tenths of the current, leaving one-tenth to pass through some other circuit. Hence, for a tenth shunt, $n=10$, and (by formula VII) the resistance of the shunt must be $\frac{1}{9}$ of the resistance of the other circuit.

Thus, for a circuit of 90 ohms' resistance, a tenth shunt must have 10 ohms' resistance. It would then carry nine-tenths of the current.

### *Voltage drop along a wire*

For uniform wires—that is, wires of the same material and size throughout their length, the resistance is proportional to the length of the wire. Thus, when a current is passing along a wire, there is a steady drop in potential along the wire. For short lengths of copper wire, this drop in potential is negligible. When the wire extends for a considerable distance, however, the drop in voltage along it should be taken into account.

*Illustrative Example*

A No. 12 B & S gauge copper wire carries a current of 20 amperes. What is the voltage drop along it?

From a table of standard resistances, we find that the resistance of No. 12-gauge wire is 0.1588 ohms per 100 feet. At the end of 100 feet, the potential difference is

$$E=IR=20\times 0.1588=3.176 \text{ volts.}$$

Hence, there is a drop of potential of approximately 3.2 volts in each 100 feet of the wire.

## THE WHEATSTONE BRIDGE

The *wheatstone bridge* is a device that makes use of the proportion between length and resistance of a wire to measure very accurately the resistance of an unknown conductor.

Fig. 8 represents one form of the wheatstone bridge. The shaded parts are strips of copper or brass of negligible resistance. $NM$ is a uniform-resistance wire connecting the ends of the outer strips. A battery at $A$ sends a steady current through the circuit. The resistor at $B$ is introduced simply to reduce the current. $G$ is a galvanometer. $R$ is a known resistance bridging a gap in the circuit. The resistance, $x$, to be measured, is connected across another gap.

The tapping key, $T$, is moved along the resistance wire, $NM$, until the

galvanometer shows no deflection. There is then no potential difference between the copper strip, $XR$, and the point, $T$, of the resistance wire, $NM$.

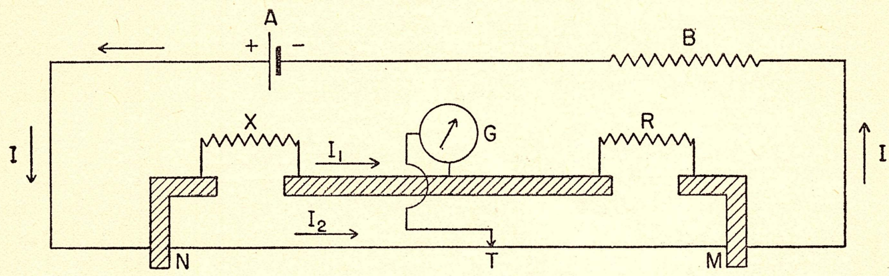



**Fig. 8**

Fig. 9 is the circuit diagram for the bridge. $R$ is the known resistance, $X$ the unknown resistance, $R_{NT}$ the resistance of that portion of the resistance wire extending from $N$ to $T$ (Fig. 8), and $R_{TM}$ is the resistance of the portion extending from $T$ to $M$.

$I_1$ is the current flowing through $NLM$, and $I_2$ is that through $NTM$. By equation II, the drop in potential from $N$ to $L$ is

$$E_{NL}=I_1X \tag{VIII}$$

and that from $N$ to $T$ is

$$E_{NT}=I_2R_{NT}. \tag{IX}$$

Since the galvanometer shows no deflection, there is no potential difference between $L$ and $T$; hence, the two $IR$-drops in equations VIII and IX are equal and may be equated thus:

$$I_2R_{NT}=I_1X. \tag{X}$$

Dividing equation X by $I_1R_{NT}$ gives

$$\frac{I_2}{I_1}=\frac{X}{R_{NT}}. \tag{XI}$$

Similarly, the potential drop from $L$ to $M$ is the same as that from $T$ to $M$. Hence,

$$I_2R_{TM}=I_1R. \tag{XII}$$

On dividing by $I_1R_{TM}$, expression XII becomes

$$\frac{I_2}{I_1}=\frac{R}{R_{TM}}. \tag{XIII}$$

Since the left-hand sides of equations XI and XIII are the same, the right-hand sides can be equated to give

$$\frac{X}{R_{NT}}=\frac{R}{R_{TM}}. \tag{XIV}$$

Multiplying both sides of equation XIV by $\frac{R_{NT}}{R}$ gives the equivalent equation,

$$\frac{X}{R}=\frac{R_{NT}}{R_{TM}}. \qquad \textbf{XV}$$

Since the wire, $NM$, is uniform throughout, the resistance of a part of it is proportional to the length of the part. Hence,

$$\frac{R_{NT}}{R_{TM}}=\frac{\text{length } NT}{\text{length } TM}.$$

Substituting in equation XV gives

$$\frac{X}{R}=\frac{\text{length } NT}{\text{length } TM}. \qquad \textbf{XVI}$$

To get an example of how formula XVI is used, let us suppose that wire $NM$ is 100 cm. long. It is found that the galvanometer shows no deflection when $TM=35$ cm., and $R=25$ ohms. Since $TM=35$ cm., we have $NT=100-35=65$ cm. Substituting in XVI,

$$\frac{X}{25}=\frac{65}{35};\quad X=\frac{65}{35}\times 25=\frac{325}{7}=46.4 \text{ ohms.}$$

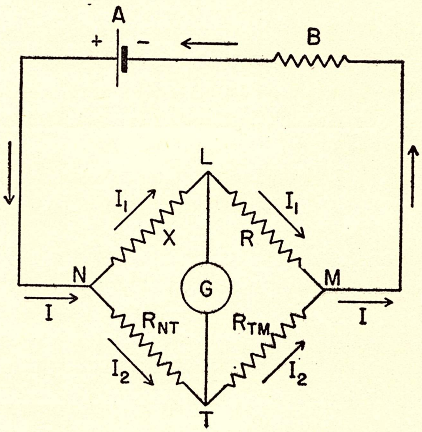


Fig. 9

## *Resistivity*

The *resistivity* of a substance is the resistance between two opposite faces of a cube of the substances with edges 1 cm. long.

If the cross-section area of a length of material remains constant, the resistance of the material is proportional to its length.

If $S$ represents the resistivity of a substance, then a bar of it $L$ cm. long and 1 cm. square has a resistance of $LS$ ohms.

If the bar has a cross-section area of $A$ sq. cm., it is equivalent to $A$ rods, each 1 sq. cm. in area, joined in parallel. If the total resistance of such a bar is $R$, then, by equation V,

$$\frac{1}{R}=\frac{1}{LS}+\frac{1}{LS}+\frac{1}{LS}+\ \ldots \text{ to } A \text{ terms}$$

$$\frac{1}{R}=\frac{A}{LS} \quad \text{or} \quad R=\frac{LS}{A}. \qquad \textbf{XVII}$$

The result is valid even though $A$ may be a fraction of a square centimeter.

*Illustrative Example*

What is the resistance of an annealed-copper bar 1 m. long, 6 cm. wide and 2 cm. thick?

The area $(A)=6\times 2=12$ sq. cm. The length $L=1$ meter $=100$ cm. The resistivity of annealed-copper at 20° C. is 0.00000172 ohm-centimeters or 1.72 microhm-centimeters.

By equation XVII, we find that the resistance is

$$R=\frac{100\times 1.72}{12}=14.3 \text{ microhms.}$$

#### TEST YOUR KNOWLEDGE OF VOLTAGE DROP AND RESISTIVITY

**14** A current of 24 amperes is drawn by a floodlight burning 9 kilowatts. What is the voltage drop across the terminal?

**15** In problem 13, how much is the voltage drop across the heating coil?

**16** A wheatstone bridge with a resistance standard 20 ohms is used to measure a smaller resistance, $R$. With $R$ in place, the bridge balances at 66.7 on a linear scale of 100. What is the resistance, $R$, in ohms?

**17** What is the resistance in ohms of a 10-meter length of copper wire 1 mm. in diameter?

**18** A 400-foot line carries a current of 8 amperes at 100 volts. What must be the diameter of the wire in the line to carry this current with a voltage drop of 2 per cent?

**19** A secondary coil contains 10,000 turns of copper wire (diameter of coil, 2 in.; diameter of wire, 0.00708 in. at 68° F.). What is the resistance in ohms?

## KIRCHHOFF'S LAWS

In 1847, G. R. Kirchhoff extended Ohm's law by making two important statements which have taken the name of their discoverer and have become known as Kirchhoff's laws. These may be stated as follows:

**a** *The algebraic sum of the currents at any function of conductors is zero;* that is, at any branch point in a circuit, there is as much current flowing away from the point as there is flowing toward it.

**b** *The algebraic sum of the electromotive forces and voltage drops around any closed circuit is zero;* that is, in any closed circuit or any closed portion of a circuit, the e.m.f. applied to the circuit is equal to the algebraic sum of the $IR$-drops in the circuit.

### *Steps in applying the laws*

By carefully following the steps given below, the student should experience no difficulty in solving circuit problems:

**a** Draw a neat diagram of the circuit, setting opposite each symbol any known values connected with it. Indicate by plus and minus the polarity of every source of e.m.f. Assign the values, $R_1$, $R_2$, $R_3$, etc., to any unknown resistances in the circuit. Assign the values, $E_1$, $E_2$, $E_3$, etc., to any unknown e.m.f.'s in the circuit.

**b** Indicate by arrows the direction of current flow around each closed portion of the circuit. Although authorities differ on this question, it is more conventional to assume that it flows from positive to negative. If there is doubt about the direction of flow in any branch, place the arrow in any convenient direction. If, in solving the problem, you obtain a negative value for any current, its direction is opposite to that indicated by the arrow, but its numerical value is the same as that calculated. Write opposite each arrow the amount of current flowing if it is known. Assign the values, $I_1$, $I_2$, $I_3$, etc., to those currents whose values are unknown.

**c** Pass around each closed portion of the circuit in the direction in which the current is assumed to flow. Place a minus sign at the first end and

a plus sign at the other end of each resistance encountered. If, in any portion of the circuit, the direction of passing is opposed to that of the current, the order of the signs must be reversed.

**d** Use Kirchhoff's first law at each branch point to write an equation involving the unknown currents. If the direction of the current is towards the branch point, its value enters the equation with a positive sign; if away from the point, with a minus sign. The student will find that not all branch points will yield a new relation between the unknown currents; hence, he should ignore those branches that do not.

**e** Use Kirchhoff's second law to write an equation for each closed portion of the circuit. To do so, proceed as follows: Begin at any point in the circuit and proceed completely around it in either direction. When an e.m.f. is passed over, write down its value preceded by a sign that is the *opposite* of the sign attached to the end of the source first encountered. When a resistance is passed over, write the value of its *IR*-drop preceded by a sign that is the *same* as the sign attached to the end first met. Set the algebraic sum of the values written down to zero upon completing the circuit. (The student will find that this process yields more equations than is necessary in the solution of the problem.)

**f** Select from the equations, written down in steps d and e, as many independent ones as there are unknown values in the complete diagram. Solve these simultaneously for the value of each unknown.

**g** Test the accuracy of the results by substituting in any of the equations that were not used in solving for the unknowns.

## Application of the laws

As an illustration of the steps given above, consider a circuit made up of three batteries and six resistances, as shown in Fig. 10. For simplicity, the internal resistance of each battery will be neglected. (To take them into account would simply insert in series with each battery a resistance equal to its internal resistance.)

Applying Kirchhoff's first law to the branch at $G$ yields the equation,

$$I_1+I_2-I_3=0. \qquad \textbf{a}$$

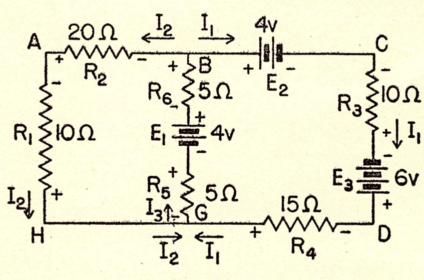



Fig. 10

Beginning at $G$ and moving around the closed circuit, $GBAHG$, we achieve, by Kirchhoff's second law,

$$-5I_3+4-5I_3-20I_2-10I_2=0,$$

which, on simplification, becomes

$$30I_2+10I_3=4. \qquad \textbf{b}$$

Circuit $GBCDG$ gives

$$-5I_3+4-5I_3-4-10I_1+6-15I_1=0,$$

or

$$25I_1+10I_3=6. \qquad \textbf{c}$$

Circuit *CDGHABC* yields

$$-10I_1+6-15I_1+10I_2+20I_2-4=0,$$

or

$$25I_1-30I_2=2. \qquad \textbf{d}$$

Selecting equations a, b, and c to use in solving for $I_1$, $I_2$, and $I_3$, we set the problem up as follows:

Solve

$$\begin{aligned} I_1+I_2-I_3&=0\\ 30I_2+10I_3&=4\\ 25I_1+10I_3&=6 \end{aligned} \qquad \textbf{e}$$

simultaneously for $I_1$, $I_2$, $I_3$. Test the results by substituting in d.

The equations grouped under e give $I_1=\frac{2}{13}$ a, $I_2=\frac{4}{65}$ a, $I_3=\frac{14}{65}$ a.

Upon substitution in equation f:

$$25\left(\frac{2}{13}\right)-30\left(\frac{4}{65}\right)\stackrel{?}{=}2$$

$$\frac{250}{65}-\frac{120}{65}\stackrel{?}{=}2$$

$$\frac{130}{65}\stackrel{?}{=}2$$

$$2=2.$$

Thus, we find that the results check.

## Three-wire distribution systems

Three-wire distribution systems are often used when the load is a combination of light and power. A three-wire system usually receives its energy from a three-wire generator, although two generators of equal voltage are sometimes connected in series. In solving circuit problems, we may consider the system as receiving its energy from two generators in either case.

### *Illustrative Example*

Fig. 11 represents a three-wire distribution system in which the loads, $L_1$, $L_2$, $L_3$, and $L_4$, consist of lamps in the number indicated. Each lamp draws one ampere. Load $M$ is a motor which draws 30 amperes. The resistance of each lead wire is indicated as a small resistance in the lead itself. Determine the amount and the direction of the current in each circuit. Determine the voltage across each group of lamps, and across the motor.

First, complete the diagram as explained on page 719. Then, application of Kirchhoff's second law to the junction at $C$ gives

$$\begin{aligned} I_1-25\text{ a}-30\text{ a}&=0\\ I_1&=55\text{ a}. \end{aligned}$$

Similarly, at junction $F$,

$$\begin{aligned} -I_2+22\text{ a}+30\text{ a}&=0\\ I_2&=52\text{ a}. \end{aligned}$$

At junction $P$,

$$-I_3+25\text{ a}-22\text{ a}=0$$
$$I_3=3\text{ a.}$$

At junction $B$,

$$I_5-20\text{ a}-55\text{ a}=0$$
$$I_5=75\text{ a.}$$

At junction $K$,

$$I_6+20\text{ a}-30\text{ a}+3\text{ a}=0$$
$$I_6=7\text{ a.}$$

At junction $G$,

$$-I_4+30\text{ a}+52\text{ a}=0$$
$$I_4=82\text{ a.}$$

To test the results at junction $J$, the law states that

$$I_4-I_5-I_6=0.$$

Substituting the values found above, we get

$$82\text{ a}-75\text{ a}-7\text{ a}=0,$$

from which the results are seen to check.

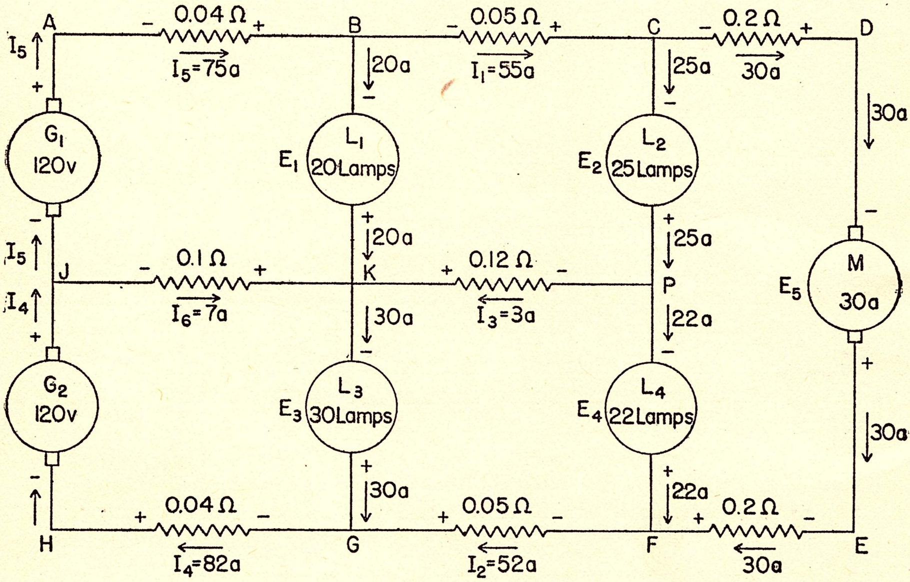



Fig. 11

Next, applying Kirchhoff's second law and passing around the closed circuit, $ABKJA$, we get

$$-75(0.04)-E_1+7(0.1)+120=0$$
$$E_1=120+0.7-3=117.7\text{ volts.}$$

For the circuit, $BCPKB$,

$$-55(0.05)-E_2-3(0.12)+E_1=0$$
$$E_2=E_1-0.36-2.75=E_1-3.11.$$

Using the value of $E_1$ just found, we get

$$E_2=117.7-3.11=114.59\text{ v.}$$

To check $E_2$, use circuit $ABCPKJA$,

$$-75(0.04)-55(0.05)-E_2-3(0.12)+7(0.1)+120=0.$$

Then $E_2 = 120 + 0.7 - 0.36 - 2.75 - 3.00 = 114.59$ v.

The circuit, *JKGHJ*, yields

$$-7(0.1) - E_3 - 82(0.04) + 120 = 0$$
$$E_3 = 120 - 3.28 - 0.7 = 116.02 \text{ v.}$$

For the circuit, *KPFGK*,

$$3(0.12) - E_4 - 52(0.05) + E_3\ (= 116.02) = 0$$
$$E_4 = 116.02 - 2.60 + 0.36 = 113.78 \text{ v.}$$

Check $E_4$ by using circuit *JKPFGHJ*:

$$-7(0.1) + 3(0.12) - E_4 - 52(0.05) - 82(0.04) + 120 = 0$$
$$E_4 = 120 + 0.36 - 3.28 - 2.60 - 0.7 = 113.78 \text{ v.}$$

Circuit *CDEFPC* gives

$$-30(0.2) - E_5 - 30(0.2) + E_4(= 113.78) + E_2(= 114.59) = 0$$
$$E_5 = 113.78 + 114.59 - 6.0 - 6.0 = 216.37 \text{ v.}$$

Check $E_5$ by using circuit *ABCDEFGHJA*:

$$-75(0.04) - 55(0.05) - 30(0.2) - E_5 - 30(0.2) - 52(0.05) - 82(0.04) + 120 + 120 = 0$$
$$E_5 = 240 - 3.28 - 2.60 - 6.0 - 6.0 - 2.75 - 3.00 = 216.37 \text{ v.}$$

As a further check, use circuit *DEFPKJABCD*:

$$-E_5 - 30(0.2) + E_4 - 3(0.12) + 7(0.1) + 120 - 75(0.04) - 55(0.05) - 30(0.2) \stackrel{?}{=} 0$$
$$E_4 + 0.7 + 120 \stackrel{?}{=} E_5 + 6.0 + 0.36 + 3.00 + 2.75 + 6.0$$
$$113.78 + 120.7 \stackrel{?}{=} 216.37 + 18.11$$
$$234.48 = 234.48.$$

### TEST YOUR KNOWLEDGE OF KIRCHHOFF'S LAWS

**20** In the circuit shown in Fig. 10, put the following values in the diagram and solve for the currents: $R_1 = 15$, $R_2 = 3$, $R_3 = 16$, $R_4 = 18$, $R_5 = R_6 = 4$ ohms; $E_1 = 8$, $E_2 = 4$, and $E_3 = 8$ volts; take the internal resistance of $E_1$ as 2 ohms, of $E_2$ as 1 ohm, and $E_3$ as 2 ohms.

**21** In the circuit shown in Fig. 11, substitute for the e.m.f.'s, number of lamps, and motor current given in the diagram the following values and solve for the unknown currents: $G_1 = 110$ v., $G_2 = 110$ v.; $L_1 = 50$ lamps, $L_2 = 45$, $L_3 = 40$, $L_4 = 30$; and for the motor, *M*, take 45 amperes. Use the same resistances shown in the diagram. (Ignore the values of the current shown in the figure. Take the number of lamps as shown. Each lamp draws one ampere.)

**22** Recompute the circuit shown in Fig. 11, but using the same e.m.f.'s, resistances, and motor current (30 amp.) shown in the diagram. For the number of lamps on each load, substitute the following: $L_1$, 18 lamps; $L_2$, 30 lamps; $L_3$, 27 lamps; and $L_4$, 25 lamps. (Ignore the values of the current shown in the figure. Each lamp draws one ampere.)

## CONDENSERS

A condenser consists of two or more metallic plates separated from each other by extremely thin layers of insulating material. The insulating material is called the *dielectric*. When a source of direct current is applied to a condenser for a brief moment and then removed, the plates of the condenser remain charged. If a conductor is later connected across the plates of the charged

condenser, a current will flow through the conductor. Thus, it is seen that a condenser has the ability to store electrical energy.

### Capacitance

The amount of energy that can be stored in a condenser depends upon the charging potential and a factor which takes into account the size of the plates, the number of plates, and the nature and thickness of the dielectric. This factor is called the *capacitance* of the condenser and is expressed in farads.

#### DIELECTRIC CONSTANT

The insulating material or dielectric used in constructing a condenser has the effect of increasing the capacitance of the condenser by a factor called the *dielectric constant* for that material. Below is a list of some of the more commonly used dielectrics:

| Material | Dielectric Constant |
|---|---|
| air | 1.00 |
| castor oil | 5.00 |
| celluloid | 4.10 |
| glass | 4.90 to 9.00 |
| lucite | 2.50 to 3.00 |
| mica | 5.75 |
| quartz | 4.75 |

#### CALCULATION OF CAPACITANCE

The capacitance of two parallel plates is given by

$$C = 0.0884\,\frac{KA}{d}, \qquad \text{XVIII}$$

where $C$ = capacitance in micro-microfarads
$K$ = dielectric constant
$A$ = area of the dielectric in square centimeters
$d$ = the distance between the plates in centimeters.

If the area is measured in square inches and the distance between the plates is measured in inches, the formula becomes

$$C = 0.2248\,\frac{KA}{d}. \qquad \text{XIX}$$

*Illustrative Problem*

A condenser is made by coating both sides of a piece of mica 10 inches square with tin foil. The thickness of the mica is 0.01 inch. The capacitance of the condenser is found by formula XIX to be

$$C = 0.2248\,\frac{5.75\times 100}{0.01} = 12{,}900 \text{ micro-microfarads}$$
$$= 0.0129 \text{ microfarads.}$$

Errors are frequently made in applying formulas XVIII and XIX to condensers consisting of a number of plates. However, no difficulties

should be encountered in such cases if one pertinent fact is kept in mind. The area, $A$, is not the area of the metallic plates but the area of the dielectric having a metallic plate on both sides of it. The following example should make the method clear:

*Illustrative Example*

A condenser is made of 100 sheets of tin foil each 5 cm. square. They are separated by sheets of lucite 0.1 mm. thick, having a dielectric constant of 3. What is the capacitance?

To separate 100 sheets of tin foil requires only 99 sheets of lucite; hence, the total area of the dielectric is

$$A=25\times 99=2475.$$

Using formula XVIII and remembering that 0.1 mm.$=$0.01 cm., the capacitance is

$$C=0.0884\,\frac{3\times 2475}{0.01}=65{,}600 \text{ micro-microfarads}$$
$$=0.0656 \text{ microfarads}$$

## Condensers

Condensers are connected either in parallel or in series.

### CONDENSERS IN PARALLEL

When condensers are joined as shown in Fig. 12, they are said to be connected in parallel.

Let $C_1$, $C_2$, and $C_3$ be the capacitance of the individual condensers, respectively, and let $C$ be the capacitance of the combination. To find $C$ in terms of $C_1$, $C_2$, and $C_3$, let a potential, $E$, be connected across the condensers. The quantity of charge in condenser $C_1$ will be

$$Q_1=C_1E,$$

and that in condensers $C_2$ and $C_3$ will be

$$Q_2=C_2E,$$
$$Q_3=C_3E.$$

The total charge will be $Q=CE$ where

$$Q=Q_1+Q_2+Q_3;$$

hence,

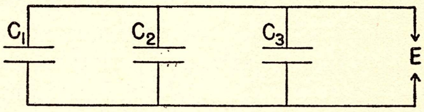



Fig. 12

$$CE=Q_1+Q_2+Q_3=C_1E+C_2E+C_3E$$
$$CE=(C_1+C_2+C_3)E$$
$$C=C_1+C_2+C_3. \qquad \text{XX}$$

Thus, it is seen that the capacitance of a number of condensers in parallel is equal to the sum of the individual capacitances.

### CONDENSERS IN SERIES

Fig. 13 represents three condensers connected in series with a potential, $E$, across the combination. Since a charge of $-Q$ on the first plate induces a charge of $+Q$ on the second, which in turn causes a charge of $-Q$ on the third, and so on, it is clear that each condenser must receive the same charge.

Let $E_1$, $E_2$, and $E_3$ represent the potential differences across $C_1$, $C_2$, and $C_3$, respectively, and $C$ the capacitance of the combination.

The quantity of charge in the combination is

$$Q = EC,$$

from which it is seen that

$$E = \frac{Q}{C}.$$

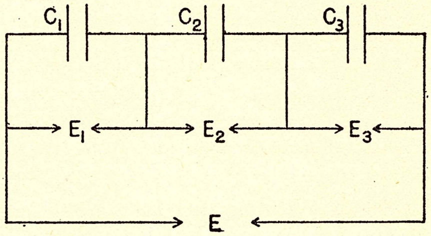


Fig. 13

Similarly, by considering each of the condensers separately, we determine that

$$E_1 = \frac{Q}{C_1}$$

$$E_2 = \frac{Q}{C_2}$$

$$E_3 = \frac{Q}{C_3}.$$

The total potential drop across the three condensers must be equal to $E$; thus,

$$E = E_1 + E_2 + E_3 \qquad \text{XXI}$$

By substituting the values for all the potentials in equation XXI, we get

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3},$$

or

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}. \qquad \text{XXII}$$

Equation XXII illustrates the fact that condensers in series combine like resistors in parallel. The reciprocal of the capacitance of the series is equal to the sum of the reciprocals of the separate capacitances.

*Illustrative Problem*

If three condensers of capacitances 0.02 $\mu$ f, 0.03 $\mu$ f, and 0.04 $\mu$ f, respectively, are connected in parallel, their combined capacitance is $0.02 + 0.03 + 0.04 = 0.09$ microfarads.

If they are connected in series, their combined capacitance is given by

$$\frac{1}{C} = \frac{1}{0.02} + \frac{1}{0.03} + \frac{1}{0.04}$$

$$= 50 + 33.3 + 25$$

$$= 108.3$$

$$C = \frac{1}{108.3} = 0.00923 \text{ microfarads.}$$

### Time constants

When a condenser is charged or discharged through a resistor, it requires a measurable length of time for the flow of current to reach a negligible value.

## TIME CONSTANT FOR CAPACITANCE

If a condenser of capacitance, $C$, is discharged through a resistor of resistance, $R$, the current flowing at time, $t$, is given by the formula,

$$I=\frac{E}{R}e^{-\frac{t}{RC}}, \qquad \text{XXIII}$$

where $e=2.718$ is the base of the natural logarithms, $C$ is measured in farads, $R$ in ohms, $t$ in seconds, $E$ in volts, and $I$ in amperes.

*Illustrative Problem*

A condenser having a capacitance of 0.01 farads is charged with a potential of 300 volts. A switch is closed, discharging the condenser through a 150-ohm resistance. What amount of current will be flowing at the end of one-half second?

Substituting in formula XXIII,

$$I=\frac{300}{150}e^{-\frac{0.5}{150\times 0.01}}=2\,e^{-0.33}.$$

From a table of powers of $e$, we find that $e^{-0.33}=0.7189$; thus,

$$I=2\,e^{-0.33}=2\times 0.7189=1.44 \text{ amperes.}$$

The *time constant* for a condenser and resistor connected in series is the length of time required for the current to drop to $\frac{1}{e}$, or approximately 37% of its initial value. From equation XXIII, we see easily that the time constant is

$$t=RC. \qquad \text{XXIV}$$

Thus, the time constant for a 0.2-microfarad condenser connected in series with a 1.5-megohm resistor is

$$t=1.5\times 0.2=0.30 \text{ seconds.}$$

## TIME CONSTANT FOR INDUCTANCE

If an inductance coil is connected in series with a resistor to a source of electromotive force, a measurable length of time is required for the flow of current to reach a steady state.

The amount of current flowing at any time, $t$, is given by the formula,

$$I=\frac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right), \qquad \text{XXV}$$

where $R$ is measured in ohms, $t$ in seconds, $L$ in henrys, $E$ in volts, and $I$ in amperes.

The time required for the current to build up to $1-\frac{1}{e}$, or approximately 63% of its final value, is called the time constant of the circuit.

From equation XXV, we see that this time constant is

$$t=\frac{L}{R}. \qquad \text{XXVI}$$

*Illustrative Problem*

An electric motor has a resistance of 10 ohms and an inductance of 5 henrys. The time constant for the motor is $t=\frac{5}{10}=0.5$ seconds. If a switch is thrown connecting the motor to an electromotive force of 120 volts, the final current (by Ohm's law) will be

$$I=\frac{E}{R}=\frac{120}{10}=12 \text{ amperes.}$$

However, at the end of 0.5 seconds, the current will be only 63% of this amount or $I=12\times 0.63=7.6$ amperes.

To find the current at the end of one second, we use equation XXV, thus:

$$I=\frac{120}{10}\left(1-e^{\frac{-10\times 1}{5}}\right)=12\,(1-e^{-2})\,.$$

From a table of powers of $e$, we find that the value of $e^{-2}$ is 0.13534 and hence

$$I=12\,(1-0.13534)=12\times 0.86466=10.4 \text{ amperes.}$$

### TEST YOUR KNOWLEDGE OF CAPACITY AND TIME CONSTANT

23 Three condensers of 0.05, 0.06, and 0.10 microfarads are connected in series. One of them is to be disconnected to increase the capacitance of the circuit. Which one will give the greatest increase?

24 A condenser having a capacitance of 300 microfarads charged to 300 volts is discharged by closing a circuit whose resistance is 500 ohms. Find the current through the circuit at the end of each fifth of a second following the discharge until the current drops below one milli-ampere.

25 A circuit has a total resistance of 120 ohms. What inductance would have the same time constant in the circuit as a condenser of 12 microfarads?

### ALTERNATING CURRENT

An *alternating current* is one which changes its direction of flow periodically. Starting in one direction, the flow builds up to a maximum, diminishes to zero, reverses direction, builds up to a maximum in the reverse direction, and returns to zero to repeat the whole sequence over again. The change from zero to maximum, to zero, to maximum in the reverse direction, and back to zero again is called a *cycle*. The number of cycles a current completes in one second is called its *frequency*.

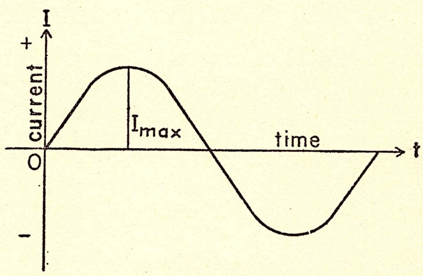



Fig. 14

One cycle of a common type of alternating current is represented graphically in Fig. 14. The horizontal axis represents the time, while the vertical axis represents the amount of

current flowing. Such a curve is described mathematically as a *sine curve* since it is represented by an equation of the form,

$$I = I_{max} \sin 2\pi ft.$$

The coefficient, $I_{max}$, is called the amplitude and is the maximum, or *peak*, value the current attains during one cycle, $f$ is the frequency, and $t$ is the time measured in seconds. Because of its wave-like appearance, the curve is also known as a *sine wave*.

## *Average and effective values*

Alternating current, or voltage, is measured in either of two ways: by finding the average value or the effective value of the current.

### AVERAGE VALUE OF ALTERNATING CURRENT AND VOLTAGE

The average value of an alternating current, or voltage, is found by averaging the instantaneous values of the current over one cycle without regard to positive or negative values.

The relationship between average and maximum values for the sine-wave form are expressed by the equations:

$$E_{av} = \frac{2}{\pi} E_{max} \qquad E_{max} = \frac{\pi}{2} E_{av} \qquad I_{av} = \frac{2}{\pi} I_{max} \qquad I_{max} = \frac{\pi}{2} I_{av}.$$

Thus, if the maximum voltage is 150 volts, the average voltage would be

$$E_{av} = \frac{2}{\pi} \times 150 = 0.637 \times 150 = 95.5 \text{ volts.}$$

If an alternating current has an average value of 10 amperes, its maximum, or peak, value would be

$$I_{max} = \frac{\pi}{2} \times 10 = 1.571 \times 10 = 15.7 \text{ amperes.}$$

### EFFECTIVE VALUES OF ALTERNATING CURRENT AND VOLTAGE

An alternating current is said to have an *effective* value of one ampere when it produces heat at the same average rate as one ampere of continuous direct current flowing through a given resistor.

The effective value of a sine-wave of current may be closely approximated by taking equally-spaced instantaneous values over one cycle and extracting the square root of their mean squared values. Thus, the effective value is called the *root-mean-square* value, and is usually abbreviated as r.m.s. The exact effective value of an alternating current, or voltage, is $\frac{1}{\sqrt{2}}$ times its maximum value. Hence the relationships between effective and maximum values are:

$$E_{eff} = \frac{1}{\sqrt{2}} E_{max} \qquad E_{max} = \sqrt{2}\, E_{eff}$$

$$I_{eff} = \frac{1}{\sqrt{2}} I_{max} \qquad I_{max} = \sqrt{2}\, I_{eff}.$$

Thus, if the maximum voltage is 150 volts, the effective voltage would be

$$E_{eff}=\frac{1}{\sqrt{2}}\times 150=0.707\times 150=106 \text{ volts.}$$

If an alternating current has an effective value of 10 amperes, its maximum value during any one cycle would be

$$I_{max}=\sqrt{2}\times 10=1.414\times 10=14.14 \text{ amperes.}$$

It is important to note that practically all meters designed for use with alternating current read the effective values of current and voltage.

Thus, if (in Fig. 1) the alternator is generating alternating current and the voltmeter reads 120 volts while the ammeter reads 2 amperes, the maximum, or peak, values of voltage and current would be

$$E_{max}=\sqrt{2}\times 120=1.414\times 120=169.7 \text{ volts}$$

$$I_{max}=\sqrt{2}\times \phantom{00}2=1.414\times \phantom{00}2=2.8 \text{ amperes.}$$

## Phase

When an alternating current flows through a circuit containing only resistance, the voltage and current go through their respective maximum and minimum values at the same instant. In such a case, the current and voltage are said to be in *phase*, or in step, with each other. This relation is represented graphically in Fig. 15.

Since, for a purely resistive circuit, the current and the voltage are always in phase, Ohm's law, as expressed in equations I, II, and III applies equally well for alternating current.

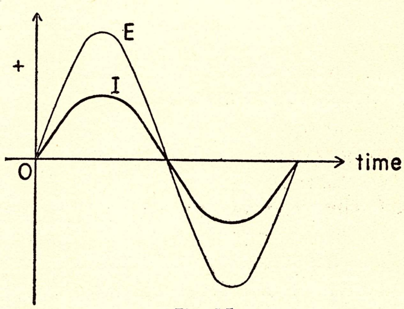


Fig. 15

As Ohm's law says nothing about maximum, average, or effective values of current and voltage, it is clear that any of these values may be used, provided only that they are properly matched. That is, maximum voltage is used to find maximum current, and effective voltage is used to find effective current.

*Illustrative Example*

An effective A.-C. voltage of 120-v. is connected to a circuit which has the resistance of 30 ohms. Find the effective value of the current that will flow.

By equation III,

$$I=\frac{E}{R}=\frac{120}{60}=2 \text{ amperes.}$$

Throughout the remaining discussion of alternating current, effective values of current and voltage will be understood unless the text explicitly states otherwise.

If an A.-C. circuit contains capacitance, or inductance, in addition to resistance, the current does not reach its maximum value at the same instant that the voltage attains its maximum value. In such cases, the voltage and the current are said to be "out of phase" with each other, and Ohm's law *does not apply*.

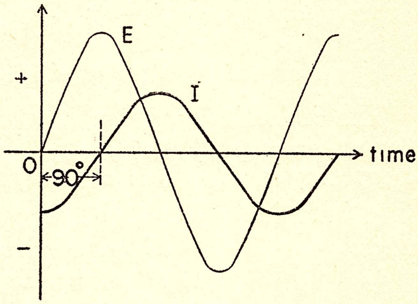


Fig. 16

The voltage-current relations for an idealized circuit containing only pure inductance are shown in Fig. 16. If time is counted from the instant that the voltage wave goes through zero from a negative to a positive value, it is seen that the current wave goes through zero from a negative to a positive value 90 time-degrees *later* than the voltage. The current is, therefore, said to *lag* the voltage by 90 degrees.

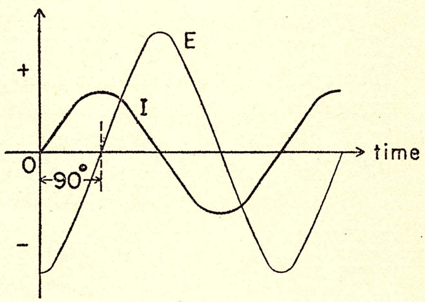


Fig. 17

Fig. 17 shows the voltage-current relations for a circuit containing only a pure capacitance. In this case, we see that the current wave goes through zero from a negative to a positive value 90 time-degrees *before* the voltage wave goes through zero from a negative to a positive value. The current is, therefore, said to *lead* the voltage by 90 degrees.

It must be clearly understood that the two cases just discussed were idealized conditions. In any actual circuit, there is always some resistance present which reduces the phase angle from that given.

## Reactance

When an A.-C. circuit contains inductance, or capacitance, the flow of current is affected by a factor other than the resistance of the circuit. This factor is called *reactance*, and is due to the presence of inductance, or capacitance, in the circuit.

Reactance is expressed in equivalent ohms, and is designated by the letter, $X$. For a given reactance at a given frequency, the current that will flow is proportional to the applied voltage; hence the equations,

$$I=\frac{E}{X}, \qquad E=IX, \qquad X=\frac{E}{I}, \qquad \text{XXVII}$$

express the relationship between voltage, current, and reactance for a purely reactive circuit—that is, for a circuit that is assumed to contain no resistance.

It is important to note that the above equations are the same as those used on pages 709 to 714, where the reactance, $X$, has taken the place of the resistance, $R$.

## INDUCTIVE REACTANCE

The *inductive reactance* of a coil is designated by the symbol, $X_L$. Its value is proportional to the inductance of the coil, and to the frequency of the A.-C. current, and is given by the formula,

$$X_L = 2\pi f L, \qquad \text{XXVIII}$$

where $X_L$ = inductive reactance in ohms
$f$ = frequency in cycles per second
$L$ = inductance in henrys.

For radio work, the same formula is used, but in that case $L$ is expressed in *millihenrys* and the frequency in *kilocycles*, or $L$ is expressed in *microhenrys* and the frequency in *megacycles*.

*Illustrative Example*

If a coil which has an inductance of 0.2 henrys and negligible resistance is connected across the terminals of a 220-v, 60-cycle alternator, how much current will flow through the coil?

By equation XXVIII, the inductive reactance of the coil will be

$$X_L = 2\pi \times 60 \times 0.2 = 75.4 \text{ ohms.}$$

The amount of current that will flow can now be determined by equation XXVII; thus,

$$I = \frac{E}{X_L} = \frac{220}{75.4} = 2.92 \text{ amperes.}$$

## CAPACITIVE REACTANCE

The *capacitive reactance* of a condenser is designated by the symbol, $X_C$. Its value depends upon the capacitance of the condenser as well as upon the frequency of the current, and is given by the formula,

$$X_C = \frac{1}{2\pi f C}, \qquad \text{XXIX}$$

where $X_C$ = capacitive reactance in ohms
$f$ = frequency in cycles per second
$C$ = capacitance in farads.

For most radio work, smaller units than those used in formula XXIX are more practical. Hence, a more convenient formula for capacitive reactance is

$$X_C = \frac{1{,}000{,}000}{2\pi f C}, \qquad \text{XXX}$$

where $X_C$ = capacitive reactance in ohms
$f$ = frequency in megacycles per second
$C$ = capacitance in micro-microfarads.

Formula XXX can also be used when the frequency is expressed in *cycles* and the capacitance is expressed in *microfarads*.

*Illustrative Example*

If a $2\mu f$ condenser were connected across a source of 300-v, 60-cycle alternating current, how much current would flow through the condenser?

By formula XXX, the capacitive reactance of the condenser at 60 cycles is

$$X_C = \frac{1{,}000{,}000}{2\pi \times 60 \times 2} = 1326 \text{ ohms, approximately.}$$

The amount of current that will flow is given by formula XXVII; thus,

$$I = \frac{E}{X_C} = \frac{300}{1326} = 0.226 \text{ amperes.}$$

From equation XXIX, it is apparent that the reactance of a condenser in the case of direct current is infinite, since, for direct current, the frequency is zero. Thus, no direct current will flow through a perfect condenser.

The fact that a condenser will allow an alternating current to flow through it but will resist the passage of direct current is frequently used in electrical circuits of all kinds. For brevity, it is said that a condenser *passes* A. C. but *blocks* D. C. Thus, in many radio circuits, for instance, certain condensers are referred to as *blocking condensers*. In such cases, it will be found that the condenser is being used to pass the radio- or audio-frequency alternating current and to prevent the passage of a direct current that would otherwise flow along the same wire.

## INDUCTIVE REACTANCE AND CAPACITIVE REACTANCE IN SERIES

When a circuit contains inductive reactance, the current lags the voltage, and when it contains capacitive reactance, the current leads the voltage. Thus, inductive reactance and capacitive reactance produce opposite effects in a circuit. The equivalent reactance of a coil and condenser connected in series is, therefore, the difference between the two reactances, and is given by the formula,

$$X = X_L - X_C. \qquad \text{XXXI}$$

### *Impedance*

In the illustrative problems of the previous sections, the resistance was, in each case, considered to be negligible. In most practical problems, however, the resistance must also be taken into consideration.

The result of combining resistance and reactance is called *impedance*,

since its effect is to impede or resist the flow of current. Impedance, like its component parts, is measured in ohms. The letter, $Z$, is usually employed to represent impedance.

When the resistance and the reactance of a circuit are known, the impedance can be calculated by the formula,

$$Z=\sqrt{R^2+X^2}, \qquad \textbf{XXXII}$$

or, since $X=X_L-X_C$, by

$$Z=\sqrt{R^2+(X_L-X_C)^2}. \qquad \textbf{XXXIII}$$

Stated in words, the impedance of a circuit is the *vector sum* of the resistance and the equivalent reactance. (See page 501.)

*Illustrative Example*

What will be the impedance of a coil and condenser connected in series to a source of 60-cycle alternating current if the coil has a resistance of 200 ohms and an inductance of 5 henrys while the condenser has a capacitance of 5 microfarads?

Fig. 18 is the equivalent-circuit diagram of the problem where the resistance of the coil appears as a simple resistor in series with the coil and condenser.

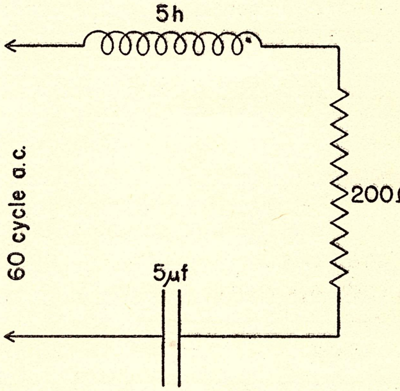


Fig. 18

The inductive reactance of the coil is, by formula XXVIII,

$X_L=2\pi\times60\times5=1885$ ohms, approx.

The capacitive reactance of the condenser is, by formula XXX,

$$X_C=\frac{1{,}000{,}000}{2\pi\times60\times5}=530 \text{ ohms, approx.}$$

The equivalent reactance is found by formula XXXI; thus,

$$X=1885-530=1335.$$

Finally, the impedance is found (by formula XXXIII) to be

$$Z=\sqrt{(200)^2+(1355)^2}=1370 \text{ ohms, approx.}$$

If, in the group of equations under XXVII, the reactance, $X$, is replaced by the impedance, $Z$, the new formulas are known as Ohm's law for alternating current. Thus,

$$I=\frac{E}{Z}, \ \textbf{XXXIVa} \qquad E=IZ, \ \textbf{XXXIVb} \qquad Z=\frac{E}{I}. \ \textbf{XXXIVc}$$

*Illustrative Example A*

If the coil and condenser described in the example above are connected in series to a 60-cycle A.-C. voltage of 1,000 volts, the current that flows is calculated by XXXIVa; thus,

$$I=\frac{E}{Z}=\frac{1000}{1370}=0.73 \text{ amperes.}$$

*Illustrative Example B*

The electromotive force required to cause a current of 1.5 amperes to flow through an impedance of 75 ohms is calculated by XXXIVb; thus,

$$E = IZ = 1.5 \times 75 = 112.5 \text{ volts.}$$

*Illustrative Example C*

If a current of 3 amperes under a pressure of 120 v is flowing through a circuit containing both resistance and reactance, the impedance of the circuit is found by XXXIVc; thus,

$$Z = \frac{E}{I} = \frac{120}{3} = 40 \text{ ohms.}$$

**TEST YOUR KNOWLEDGE OF IMPEDANCE**

**26** (a) What is the reactance in ohms of a condenser whose capacitance is 0.00007 farads, if the current has a frequency of 60 cycles per second? (b) Is this an inductive or a capacitive reactance?

**27** What is the reactance of a coil with an inductance of 2 microhenrys and a condenser of 3 microfarads, connected in series, if the current has a frequency of 75 kc.?

### Resonant circuits

Fig. 19 represents a coil, a resistor, and a condenser connected in series to a variable-frequency alternator. The magnitude of the alternating current that flows depends upon the electromotive force of the alternator and the impedance of the circuit, which, in turn, depends upon the frequency of the current.

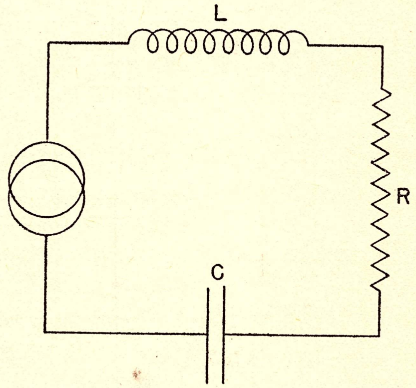


Fig. 19

Suppose that the electromotive force is kept at a constant value but that the frequency of the current is gradually changed from a low frequency to a high one. For any given combination of coil, resistor, and condenser, a certain frequency will be found for which a greater amount of current will flow around the circuit than will flow for any other frequency. The frequency at which this greatest flow of current is obtained is called the *resonant frequency* of the circuit.

A study of equations XXVIII and XXIX clearly explains the cause of the phenomena described above. From XXVIII, it is seen that the reactance of the coil *increases* with the increase in frequency, while equation XXIX

shows that the reactance of the condenser *decreases* as the frequency increases. Thus, if the frequency starts at a low value and gradually increases, a frequency will eventually be reached at which the reactances of the coil and of the condenser are equal.

By equation XXXIII, the impedance has its lowest value when $X_L = X_C$; hence, the flow of current will be a maximum when $X_L = X_C$.

Clearly then, at the resonant frequency, the inductive reactance is equal to the capacitive reactance, and impedance is equal simply to the resistance of the circuit.

To find the resonant frequency for a given value of capacitance and inductance, set the values of $X_L$ and $X_C$ from equations XXVIII and XXIX equal. Thus,

$$2\pi fL = \frac{1}{2\pi fC}.$$

Multiplying both sides by $f$ yields

$$2\pi f^2 L = \frac{1}{2\pi C}$$

which, on division by $2\pi L$, becomes

$$f^2 = \frac{1}{4\pi^2 LC}.$$

Extracting the square root of both sides gives

$$f = \frac{1}{2\pi\sqrt{LC}} \qquad \text{XXXV}$$

where $f$ = frequency in cycles per second
$L$ = inductance in henrys
$C$ = capacitance in farads.

Formula XXXV is not convenient to use in radio calculations, as the units of inductance and of capacitance are too large. A more suitable formula is

$$f = \frac{1000}{2\pi\sqrt{LC}}, \qquad \text{XXXVI}$$

where

$f$ = frequency in megacycles per second
$L$ = inductance in microhenrys
$C$ = capacitance in micro-microfarads.

Formula XXXVI can also be used when

$f$ = frequency in kilocycles per second
$L$ = inductance in microhenrys
$C$ = capacitance in microfarads.

*Illustrative Example*

If a 30-microhenry coil is connected in series with a 270-microfarad condenser, what is the resonant frequency of the circuit?

The units given suggest the use of formula XXXVI. Thus,

$$f = \frac{1000}{2\pi\sqrt{30 \times 270}} = 1.76 \text{ kilocycles},$$

is the resonant frequency.

### TEST YOUR KNOWLEDGE OF FREQUENCIES

28 At what frequency does a coil of 1 millihenry have a reactance equal to a condenser of 1 millifarad?

29 A circuit contains a condenser of 120 microfarads. The total resistance of the circuit is 60 ohms. Taking its total inductance as 0.1 henrys, find the e.m.f. which must be delivered by a generator to maintain a current of 2 amperes in the circuit at each of the following frequencies: 30, 45, 60, and 90 cycles per second.

## RADIO APPLICATIONS OF OHM'S AND KIRCHHOFF'S LAWS

Every principle that has been considered in the preceding sections of this article is a fundamental principle of electricity. Thus, each of them has many applications in every branch of electricity.

It is our purpose in this final section to indicate how these principles are applied in the field of radio. This may be done by considering a few specific radio problems to show how these principles are used in their solutions.

### *Bleeder resistors*

Resistors are often connected across the output terminals of the power supply of radio receivers and of transmitters. The purpose of these resistors, known as *bleeder resistors*, is to bleed off a constant value of current, or to act as a fixed load on the power supply. The bleeder resistor thus improves the regulation of the power supply, and the voltage is maintained at a more constant value, should the remaining load conditions change. The bleeder resistor also prevents the filter condenser from being damaged by a sudden rise in voltage, should the external load drop to zero.

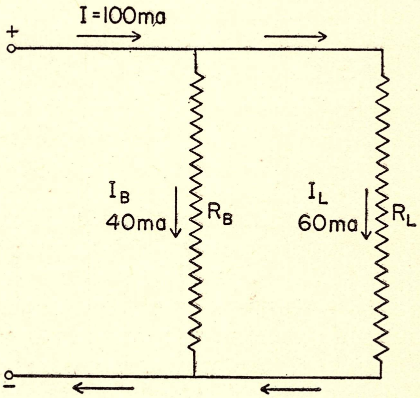


Fig. 20

*Illustrative Example*

A particular power supply will safely deliver a current of 100 milliamperes at 300 volts. If the receiver requires only 60 milliamperes, what value should the bleeder resistor have so as to bleed off 40 milliamperes of the current?

Fig. 20 is the equivalent-circuit diagram of the problem where $R_L$ is a resistance which simulates the load, and $R_B$ is the bleeder resistor.

By equation XXVII,

$$R_B=\frac{E}{I_B}=\frac{300}{0.04}=\frac{30{,}000}{4}=7{,}500 \text{ ohms.}$$

Thus, the value of the bleeder resistor should be exactly 7,500 ohms.

To check the results and to be sure that the total current is exactly 100 milliamperes, first use the same equation to find the resistance of the load; thus,

$$R_L = \frac{E}{I_L} = \frac{300}{0.06} = \frac{30,000}{6} = 5,000 \text{ ohms.}$$

Next, the total equivalent resistance of the load and bleeder is found by equation XXXI; thus:

$$\frac{1}{R} = \frac{1}{R_B} + \frac{1}{R_L} = \frac{1}{5,000} + \frac{1}{7,500} = \frac{5}{15,000}$$

$$R = \frac{15,000}{5} = 3,000 \text{ ohms.}$$

Finally, the total current is found (by equation XXIX) to be:

$$I = \frac{E}{R} = \frac{300}{3000} = 0.1 \text{ amperes} = 100 \text{ milliamperes.}$$

## *Voltage dividers*

The plates and screens of the various tubes in a radio receiving set all require a certain amount of direct current at a particular voltage. The voltage required on the plate of one tube may not be the same as that required on another, and the voltage required on the screen-grid of a tube is usually not the same as that required on its plate. Hence, the power supply of the receiver must furnish direct current of various voltages.

In most power supplies, these various voltages are all obtained from a single source of direct current, which has a voltage as high as, or higher than, that required by any one of the tubes. The other voltages, all of which are smaller than the original, are then obtained by various arrangements of resistors.

One arrangement of resistors that will supply several D.-C. voltages is called a *voltage divider*, a resistor with several taps along its length, connected across the terminals of the output of the power supply; it sometimes takes the form of several resistors connected in series across the output terminal. In most cases, the voltage divider also acts as a bleeder resistor.

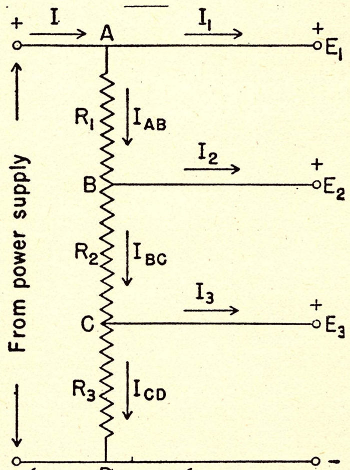


Fig. 21

Fig. 21 is the equivalent-circuit diagram of a voltage divider that will furnish three voltages, $E_1$, $E_2$, and $E_3$, and, in addition, a bleeder current of $I_{CD}$ through the resistor, $R_3$.

If $I_1$, $I_2$, and $I_3$ are the three load currents and if $I_{AB}$, $I_{BC}$, and $I_{CD}$ are the currents through the resistors, $R_1$, $R_2$, and $R_3$, respectively; then by Kirchhoff's first law

$$I=I_1+I_{AB} \qquad \text{XXXVII}$$

$$I_{AB}=I_2+I_{BC} \qquad \text{XXXVIII}$$

$$I_{BC}=I_3+I_{CD} \qquad \text{XXXIX}$$

Substituting the value of $I_{BC}$ from equation XXXIX in equation XXXVII yields

$$I_{AB}=I_2+I_3+I_{CD}. \qquad \text{XL}$$

Placing the value of $I_{AB}$ in equation XXXVII yields

$$I=I_1+I_2+I_3+I_{CD}. \qquad \text{XLI}$$

Thus, it is seen that the total current is equal to the sum of the three load currents plus the bleeder current, $I_{CD}$.

The voltage, $E_1$, is the output voltage of the power supply. The voltage drop from $A$ to $B$ is $E_1-E_2$. The value of $R_1$ is found (by Ohm's law) to be

$$R_1=\frac{E_1-E_2}{I_{AB}}=\frac{E_1-E_2}{I_2+I_3+I_{CD}}. \qquad \text{XLII}$$

In a similar manner, $R_2$ and $R_3$ are found to be

$$R_2=\frac{E_2-E_3}{I_{BC}}=\frac{E_2-E_3}{I_3+I_{CD}} \qquad \text{XLIII}$$

$$R_3=\frac{E_3}{I_{CD}}. \qquad \text{XLIV}$$

*Illustrative Example*

The output of a power supply can safely supply 150 ma at 300 v direct current. A radio receiving set requires 300 v at 80 ma and 250 v at 10 ma for the plates of the tubes and 100 v at 6 ma for the screen-grids. What must be the resistance of each section of a voltage divider that will supply the voltages required and will, at the same time, act as a bleeder for the receiving set?

From the statement of the problem, $E_1=300$ v, $E_2=250$ v, $E_3=100$ v, $I_1=80$ ma, $I_2=10$ ma, $I_3=6$ ma, and $I=150$ ma.

The amount of bleeder current is first found (by equation XLI); thus:

$$I=I_1+I_2+I_3+I_{CD}$$

$$150=80+10+6+I_{CD}$$

$$I_{CD}=150-96=54 \text{ milliamperes.}$$

The resistance, $R_3$, is (by equation XLIV):

$$R_3=\frac{E_3}{I_{CD}}=\frac{100}{0.054}=1{,}851.9 \text{ ohms.}$$

The resistances, $R_2$ and $R_1$, are found in turn from equations XLIII and XLII.

$$R_2=\frac{E_2-E_3}{I_3+I_{CD}}=\frac{150}{0.06}=2{,}500 \text{ ohms}$$

$$R_1=\frac{E_1-E_2}{I_2+I_3+I_{CD}}=\frac{50}{0.07}=714.3 \text{ ohms.}$$

### *Tube filaments in series*

A large number of different kinds of radio tubes have the same voltage rating for their filaments, or heaters. However, the filaments of two such tubes cannot be connected in series across a voltage of double their voltage rating unless the current ratings of the two tubes are also the same. Thus, two tubes with a filament rating of 6.3 volts cannot always be connected in series to a 12.6-volt source.

*Illustrative Example*

The heaters of a 6K8 tube and a 6V6 tube are connected in series to a 12-6 volt supply of current. Calculate the current flowing through the filaments and the voltage drop across each.

The filament of a 6K8 tube has a rating of 6.3 volts at 0.3 amperes, while the rating of a 6V6 tube is 6.3 at 0.45 amperes.

The resistance of each filament can be found by Ohm's law; thus $R_1=\frac{E}{I_1}=\frac{6.3}{0.3}=21$ ohms is the resistance of the 6K8 tube, and the resistance of the 6V6 tube is $R_2=\frac{E}{I_2}=\frac{6.3}{0.45}=14$ ohms. As the filaments are connected in series, their combined resistance is $R=R_1+R_2=21+14=35$ ohms.

The same amount of current flows through each filament. It is $I=\frac{E}{R}=\frac{12.6}{35}=0.36$ amperes. The voltage drop across the 6K8 tube is $E_1=IR_1=0.36\times 21=7.56$ volts. The voltage drop across the 6V6 tube is: $E_2=IR_2=0.36\times 14=5.04$ volts.

This example shows why two tubes cannot always be connected in series even though their voltage ratings are the same. It is to be noted that the filament with the lower current rating draws too much current and has too great a voltage drop across it, while the filament with the higher current rating does not draw enough current and has too small a voltage drop across it.

Two radio tubes with the same filament voltage rating can be connected in series by shunting a resistor across the tube, thus:

The 6K8 tube has a filament current rating of 0.3 amperes while that of the 6V6 tube is 0.45 amperes. The difference is $0.45-0.3=0.15$ amperes. To find the value of the resistor to be shunted around the 6K8 tube, it is necessary only to divide 0.15 amperes into the desired voltage. Thus,

$$R=\frac{6.3}{0.15}=42 \text{ ohms.}$$

If a resistance of 42 ohms is shunted around the 6K8 tube with a filament resistance of 21 ohms, the combined resistance will be

$$\frac{1}{R}=\frac{1}{42}+\frac{1}{21}=\frac{3}{42} \text{ or } R=\frac{42}{3}=14 \text{ ohms.}$$

As the filament resistance of the 6V6 tube is also 14 ohms, it is clear that both tubes will have a voltage drop of 6.3 across their filaments.

# · MILITARY GUNNERY ·

By Sebastian B. Littauer, D.Sc.

WHEN one fires a rifle at a target only 1000 yards away, one depends on the accuracy of the rifle and on a simple rule for sighting. With little instruction, but with a little practice, one can learn how to adjust the sight so as to hit the mark from a given distance. Such gunnery, still an important part of modern military action, requires no recourse to theoretical principles.

It is a vastly different problem, however, when the target is not visible to the gunner, and when its distance is some thousands of yards from the gun. In naval battle, the enemy ship may be even below the horizon. Modern military gunnery is the result of the application of many fields of science. Official training manuals on the operation of a single field piece cover several hundred pages. In PRACTICAL MATHEMATICS, we are selecting a few basic principles on the use of mathematics in *gunnery*—the practical application of ballistics to artillery fire in order to place a projectile on a target. After mastering the material presented here, you will be better able to appreciate the treatment in the larger works.

## PREPARATION OF FIRE

Suppose it be our job to fire a gun: our objective is to "lay" the projectile on the target. *Laying* means giving the piece the proper direction and inclination so as to produce the desired trajectory; that is, one which passes through the target. (The *trajectory* is the path taken by the center of the projectile in flight.) The target may be visible and the laying *direct*, but more often the target is invisible and the laying is *indirect*, that is, upon a fixed object, other than the target, called the *aiming point*. In either case, we are confronted by a primary problem of finding the *range* (the distance from the piece to the target). While the target and piece are not always at the same elevation, we shall assume the same elevation for the scope of this study in order to simplify the presentation and not confuse the reader unnecessarily.

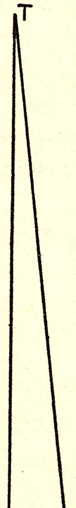
### *Range finding*

Naval guns and field artillery are equipped with *range finders*. We shall consider one schematically and study the limits of their accuracy.

In Fig. 22, $P_L$ and $P_R$ are precisely ground pentagonal prisms; $M_L$ and $M_R$ mirrors set at right angles; $L$ an adjusting lens (set perpendicular to axis $A_1A_2$) which can be moved along the axis of the range finder; $O$ the objective lens through which one can view the reflection of the rays coming from $P_L$ and $P_R$ as they are reflected off the crossed mirrors, $M_L$, $M_R$; $S$ the adjusting device, scale, and lens through which the scale is read, by which $L$ is moved left or right.

The operation is as follows: A ray of light which enters face $F_H$ of $P_L$ or $P_R$ is by two reflections sent out of face $F_P$ at right angles to the entering direction. (The reader might find it an interesting exercise to prove this.) The range finder is rotated until the line of sight from the target to $P_L$ is perpendicular to axis $A_1A_2$. The observer knows this to be the case when the vertical axis of the target is superimposed on the vertical line in $M_L$, which lies in a vertical plane and is at right angles to $A_1A_2$. Note that the lower half of $P_L$ is blacked over so that only the upper half of the target is visible in $M_L$.

Rays from $T$ enter $P_R$ also, but not at right angles to $A_1A_2$. On $F_P$ of $P_R$, the upper half is blacked out so that rays emerge only from its lower half. These rays, when reflected off $M_R$ without adjustment of $L$, are seen as in Fig. 22. They present, together with the reflection off $M_L$, a broken image of $T$. By moving $L$ to the left or the right, we can bring the rays emerging from $P_R$ into alignment with those from $P_L$ so that one sees through $O$ a whole image of $T$ reflected off $M_L$ and $M_R$. The amount of adjustment of $L$ necessary is determined by $R$, the distance of $T$ from $A_L$, which is, of course, the range. Through the eyepiece at $S$, there is a scale which is calibrated to the range, or from which reading the range can be obtained by reference to a table.



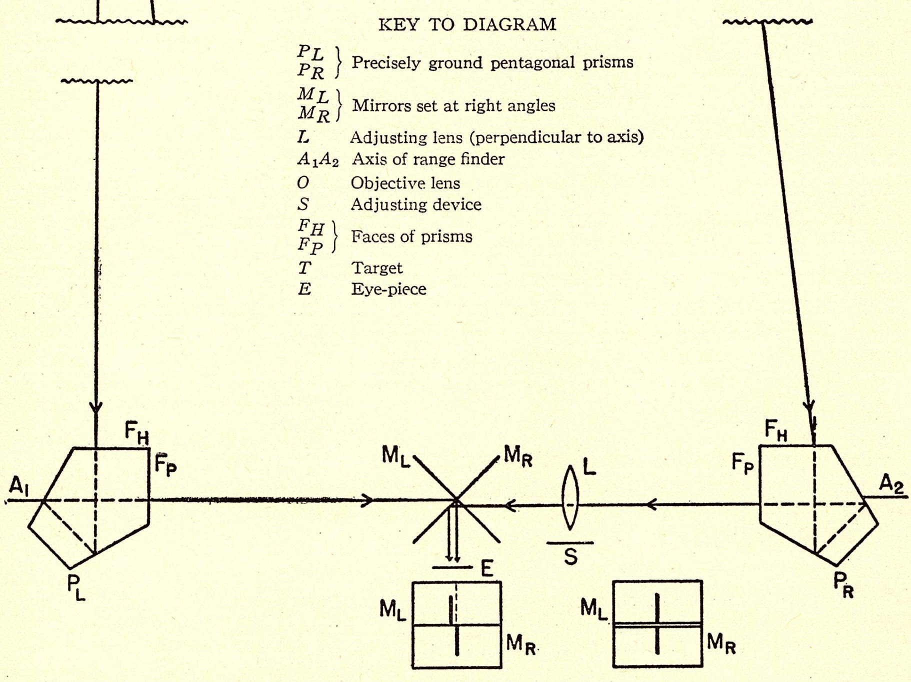




Fig. 22

Obtaining the distance, $R$, is mathematically very simple. (The difficulties in the range finder lie in making an optically accurate system; not in its theory.) We wish to find $R$: we know $B$. What else can we know? Obviously $\theta$, if the adjustments of $L$ can be made accurate enough. From elementary trigonometry, it follows that $R=\dfrac{B}{\tan\theta}$.

Simple enough. If the range finder measures $\theta$, we can compute $R$. A truly clever instrument maker, however, goes beyond this; he calibrates scale $S$ so as to read $R$ directly, dependent on the position of $L$ necessary to produce the whole image of $T$ in the reflection off $M_L$ and $M_R$.

You are not entirely satisfied with this range, and you are still wondering how far off the range may be. Here is where a little mathematics tells the story—of course, after we agree that the instrument itself is accurate. When you gaze at Fig. 23, you may wonder how small an imperfection in the image your eye detects. Physicists and psychologists say that we can distinguish between two points if the distance between them subtends an angle of fifteen seconds at the eye. Now if the eyepiece, $E$, magnifies 30 times, an angle of $\frac{1''}{2}$ will be blown up to $15''$. This means that we should detect an error of $\frac{1''}{2}$ in the measurement of $\theta$. We can use a little calculus to answer this question: If, in operating our range finder, we should make an error, plus or minus, of $\frac{1''}{2}$ in determining $\theta$, what error do we make in $R$, when $R$ is 1000 yards?

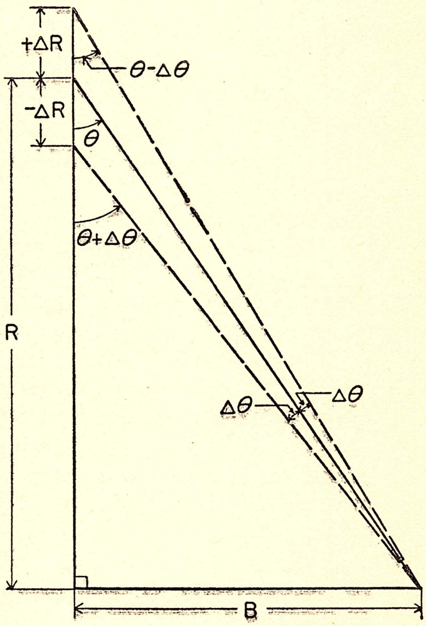


Fig. 23

We know that

$$\tan\theta=\frac{B}{R}=\frac{1}{1000}=0.001$$

$$\theta=3.099\text{ minutes }(3.099')$$

For angles less than 30 minutes (30′), sine and tangent agree in their first three significant figures, and both of these numbers agree in the same number of figures with the measure of $\theta$ expressed in radians. Hence, if $\tan\theta=0.001$, $\theta=0.001$ radian.

Then

$$\tan\theta \cong \theta = \frac{B}{R}$$

$\cong$ denotes "equals approximately".

$\theta < 30'$, 3-figure accuracy

(that is, there is equality to three figures).

Let the error in $\theta = \frac{1''}{2} = 0.0000024$ radian. We find the error in $R$ by letting $\theta$ vary by 0.0000024 and finding out by how much $R$ varies. This sounds like finding a derivative and the dependent differential. The derivative of $R$ with respect to $\theta$ if $\theta = \frac{B}{R}$ is

$$D_\theta R = \frac{-B}{\theta^2}.$$

If $\theta$ varies by $\Delta\theta$, by how much does $R$ vary? It has been shown that, if $\theta$ is small $\left(\text{say, } \frac{1}{1000} \text{ of } \theta\right)$ then

$$|\Delta R| \cong |D_\theta R \times \Delta\theta|.$$

Hence, if $B=1$ yard, $\theta=0.001$ radian, and $\Delta\theta=0.0000024$ radian, then $R=1000$ yards and

$$\left|\Delta R\right| \cong \left|-\frac{B}{\theta_2}\times \Delta\theta\right| = \left|-\frac{1}{0.000001}\times 0.0000024\right|$$
$$\cong |-2.4|$$
$$\cong 2.4 \text{ yards}$$

That is, when we are operating this type of range finder on a target 1000 yards away, efficient reading and adjustment should give the range as greater than 997.5 and less than 1002.5, or $1000 \pm 2.5$ yards.

### TEST YOUR ABILITY TO USE RANGE FINDERS

**1** Make a table of errors for the above range finder for ranges of 2,000, 4,000, 6,000, 8,000, and 10,000 yards.

**2** Make a table of errors for a horizontal 9-foot range finder for ranges of 5,000, 10,000, 15,000, and 20,000 yards. (Assume error in $\theta$ same as above.)

**3** Two different operators, A and B, have limits of visual acuity of 8″ and 12″ respectively. If each uses a range finder whose eyepiece magnifies 24 diameters, compute for each operator the error in $\theta$, both in seconds of arc and in radians.

**4** Under the same conditions as in problem 3, compute a table of errors for operator A with a 6′ range finder, at ranges of 4,000, 8,000, 12,000, and 16,000 yards.

**5** Under the conditions of problem 3, compute a table of errors for operator B, using a 15″ range finder at ranges of 12,000, 16,000, 24,000, and 32,000 yards.

### DIRECTION

A range finder can be used for both direct and indirect laying. More often, we are confronted with the task of indirect laying. In the first place, we sight at an aiming point, $P$, other than the target, $T$,

where in general the range of $P$ is the same as the range of $T$. Then we sight from an observation post, $O$, at some convenient distance from the gun, $G$ (Fig. 24).

In order to operate, we must find the *firing angle* (the angle subtended at the piece, $G$, by $TP$, measured clockwise). To do this by observations taken at $O$, we resort to a little geometric subterfuge.

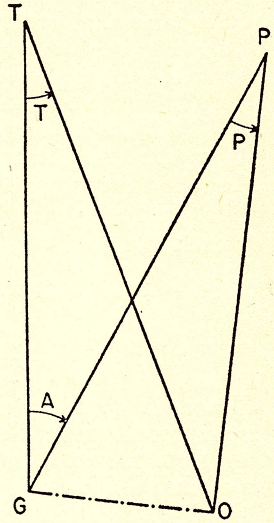


Fig. 24

Examine Fig. 24 and note that the angles, $GTO$ and $GPO$, characterize the configuration, $GOPT$. They are called *target offset* and *aiming point offset*, respectively.

Angle $T$ in Fig. 25 is readily obtained from

$$\sin T = \frac{OQ}{OT}.$$

$OQ$ cannot readily be measured, but $\angle TGO$ can.

$$\angle TGO \cong \angle(180° - \angle GOT).$$

Hence, since $\quad OQ = OG \sin TGO$

$$\sin T \cong \frac{OG \sin TGO}{OT}.$$

A similar relationship can be found for $P$.

In general, since $OG$ is small compared with $OT$ or $OP$, $T$ or $P$ in radians can replace $\sin T$ and $\sin P$, respectively. Because of the frequency with which small angles are used and because of the need for a convenient measure for angles and arcs, the *mil* is used in military gunnery. (See page 391.)

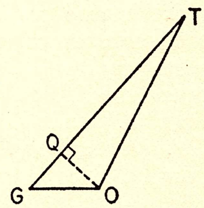


Fig. 25

Suppose now that in finding

$$T = \frac{OG}{OT} \sin TGO,$$

$OG = 100$ yd., $OT = 3000$ yd., and $\sin TGO = 0.3$

Then $T = \frac{100}{3000} \times 0.3 = \frac{1}{100}$ radian, or 10 mils.

It is fairly safe to take $\frac{OT}{1000}$ since $OT$ is the range, $R$, expressed, usually, in thousands of yards. Then, when $OG$ is surveyed, the range taken, and the factor, $\sin TGO$, known,

$$T = \frac{OG \sin TGO}{\frac{R}{1000}} \text{ mils} = 1000 \frac{OG}{R} \sin TGO \text{ mils}$$

In practice, if $OGT$ is greater than 75°, $\sin OGT$ may be taken as unity.

The advantage of the mil unit is this: in gunnery, one of the most frequent problems is that of determining offsets, widths, or ranges in the simple relation of the range finder or in points related as $T$, $G$, and $O$. Then, as in Fig. 26,

$$T \cong \frac{W}{R} \text{ mils.}$$

### TEST YOUR ABILITY TO COMPUTE DIRECTION

6 A ship's length at 7000 yards appears 50 mils wide (subtends an angle of 50 mils at the observer). How long is the ship?

7 In preparing for a 5000-yard range, we desire to set up three possible observation posts at 50, 100, and 150 yards from the gun. What is the offset in each case?

8 Draw up a table for converting angles up to 90° (by 10° differences) into mils.

9 Prepare a similar table up to 60′.

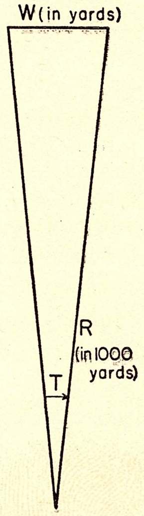


Fig. 26

Another advantage may be obtained by referring to all angles, large or very small, in terms of a single unit. For example,

$$1° = \frac{6400}{360} = \frac{160}{9} = 18 \text{ mils, approximately}$$

$$30' = \frac{160}{9 \times 2} = \frac{80}{9} = 9 \text{ mils, approximately}$$

Hence, in changing the *range setting* (angle at which the piece is set in order that the projectile will fall at a given range from the gun), it is much simpler to refer to lowering the setting 507 mils by 27 mils, than to refer to lowering 28°30′ by 1°31′. Scales and verniers calibrated to mils are much simpler to read than are degree-minute scales and verniers.

Now let us find the firing angle, $A$, assuming that we have the data for the offsets, $T$ and $P$. In the case depicted in Fig. 27, the aiming point is said to be in front (when $T$ and $P$ are on the same side of $GO$) and the piece is said to be on the left (when the piece is on the left of $OT$, observer facing $T$). In order to carry all operations from $O$, pass through $O$ lines $OT'$ and $OP'$ parallel to $GT$ and $GP$, respectively. From theorems on parallel lines, it follows directly that

$$T = TOT', \quad P = POP',$$

and $$T'OP' = TGP = A,$$

the firing angle; and since

$$TOP' = TOP + P = T + A,$$

letting $TOP$ be denoted by $M$,

$$A = M + P - T.$$

$M$ can be measured from $O$ as can $T$ and $P$, so that $A$ can be determined by observations principally made at $O$.

If we define an aiming point "rear" when $T$ and $P$ are on opposite sides of $OG$, and the piece "right" when on the right of $OT$, observer facing $T$, there are four different cases from which to find $A$.

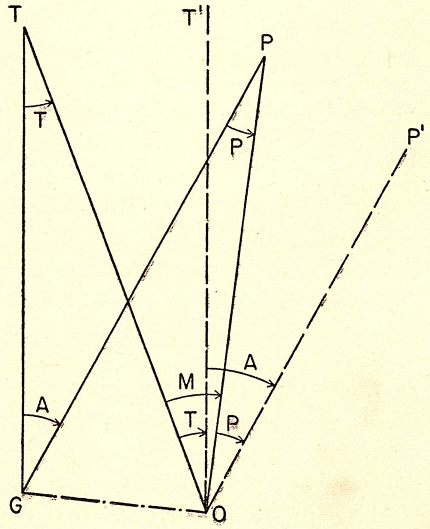


Fig. 27

### TEST YOUR ABILITY TO FIND FIRING ANGLES

**10** Fill in the following table for the remaining cases.

| Aiming Point | Piece | Firing Angle |
|---|---|---|
| Front | Left | $M+P-T$ |
| Front | Right | ......... |
| Rear | Left | ......... |
| Rear | Right | ......... |

The following points are given in terms of azimuths, back azimuths, and ranges (distances). All azimuths and back azimuths are taken with respect to point $O$. All ranges are measured from $O$. *Azimuth* of any point $P$ with respect to $O$ is the true bearing of $P$ as observed from $O$. That is, it is the angle measured from the true North (or $Y$-axis direction) clockwise to the line joining $P$ to $O$. *Back azimuth* equals azimuth plus 180°.

In each of the following problems find the firing angle, $A$.

**11** Azimuth $T$ $=000$ (means 0° measured as defined)
Azimuth $P$ $=013$ (means 13°)
Back azimuth $G=270$ (means 270° measured as defined. Hint: azimuth $G=90°$)
Range to $T=3000$ yards or units
Range to $P=4125$ yards or units
Range to $G=1000$ yards or units

**12**

| | Azimuth | Back Azimuth | Range |
|---|---|---|---|
| $T$ | 000 | ........ | 3000 |
| $P$ | 116 | ........ | 2330 |
| $G$ | ........ | 270 | 500 |

**13**

| | Azimuth | Back Azimuth | Range |
|---|---|---|---|
| $T$ | 000 | ........ | 4000 |
| $P$ | 007 | ........ | 5100 |
| $G$ | ........ | 304 | 1800 |

(Perhaps graphic solutions will be helpful.)

## THE TRAJECTORY

Observations and calculations preparatory to firing a gun are valid only if we know the path which a projectile pursues. Determining a projectile's *trajectory* (path traveled by center of projectile), is a rather involved problem.

### *Variables influencing the trajectory*

Having set the axis of the bore at a given angle and being ready to fire the gun, we desire to know what path the projectile will take and how far it will travel. Let us first determine the shape of the trajectory.

The velocity given the projectile along its trajectory is the first variable, $V_0$. A velocity is composed of a speed and a direction. While it is customary to let $V_0$ be expressed in feet per second, it is necessary to specify $\phi$, the angle of elevation, the angle which the axis of the bore makes with the horizontal direction (we assumed that we should consider only horizontal ranges) if we are to specify the initial velocity completely.

The projectile moves through air which may affect the trajectory, but let us skip that. Assume that the projectile is traveling in a

vacuum, and that the only influences on it are $V_0$, $\phi$, and, lest we forget, the effect of gravitation. We may, for purposes of inquiry, rule out the air, for a vacuum is theoretically possible, and physically approachable, but we cannot dispose of gravitation. If you recall some of your physical experimentation, you will recognize the effect that gravitation has on a body; namely a downward acceleration of 32.2 feet per second; that is, the acceleration given the body is produced by a force equal to the body's weight, $W$. We assume further that this effect is constant and denote it by $g$; and that the motion of one point in the projectile, its physical center, traces the path for which we are looking.

Now, take stock of the situation and look at Fig. 28.

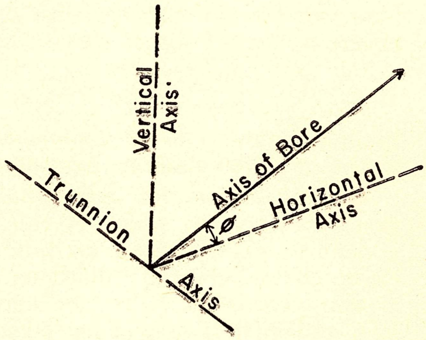


Fig. 28

The bore can be swung in a vertical plane about its trunnions; that is, the axis of the bore can swing from the horizontal upward, about a horizontal axis which is perpendicular to the axis of the bore. For accurate fire, the trunnion axis must be truly horizontal, and deviation in this respect must be accounted for. If all our assumptions are fulfilled, the trajectory will lie wholly in the vertical plane determined by the horizontal and vertical axes, respectively. Now we must find that trajectory.

The particle, $A$ (Fig. 29), is acted on by vertical force $W$ only, although it possesses also an initial velocity as indicated. Remember, however, that, according to our physical experience as expressed by Newton, a body maintains a given velocity unaltered unless affected by a force. The velocity, $V_0$, merely persists but it has no force equivalent.

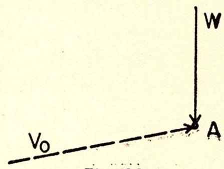


Fig. 29

In order to study the movement of $A$, we must derive some relationship between time of flight and coördinates of position. Let the axes of coördinates originate through the initial point of flight as in Fig. 30, and let $(x, y)$ designate the coördinates of $A$, at any time, $t$. What do we know about $(x, y)$ that puts them into some equation? We know one fact (namely, that $a = g$) but how do $x$ and $y$ appear here? Since $g$ is a constant, the $x$ and $y$ must be contained in $a$, and $a$, a vertical acceleration, means the rate of change of vertical velocity, $v_y$, with time, $D_t v_y$ or $\frac{dv_y}{dt}$. Hence, $\frac{dv_y}{dt} = 32.2$ feet per second per second; but $v_y$ itself is $\frac{dy}{dt}$, whence one of our variables appears, and we apparently have $\frac{d^2y}{dt^2} = 32.2$ per second per second.

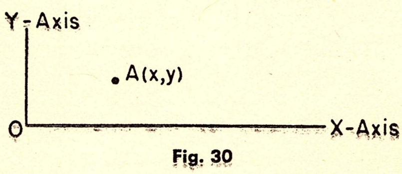


Fig. 30

If $y$ is measured positively upward while the acceleration is downward, then

$$\frac{d^2y}{dt^2} = -32.2 \text{ feet per second per second.}$$

What has happened to variable $x$? Nothing,—it merely had not yet emerged. Force $W$ is vertically downward and has no effect (component) at right angles to itself; hence, in the horizontal direction there is no force acting and therefore no acceleration, or, better, the horizontal acceleration is zero. This is denoted by

$$\frac{dv_x}{dt} = \frac{d^2x}{dt} = 0.$$

Now, we have a mathematical expression of the instantaneous conditions governing the motion of $A$:

$$\frac{d^2y}{dt^2} = 32.2 \; \frac{\text{ft.}}{\text{sec.}^2} \qquad \frac{dv_y}{dt} = 32.2 \qquad \text{or} \qquad \frac{d^2x}{dt^2} = 0 \qquad \frac{dv_x}{dt} = 0 \qquad \textbf{Ia}$$

From these relations, we should find direct relations between $x$ and $y$; that is, one or more equations limiting the paired values that $(x, y)$ may take on.

We have a pair of differential equations to solve, in which $v_0$ must have a definite value (speed and direction). Since our differential equations are in $x$- and $y$-variables, why not express $v_0$ in $x$ and $y$, if it is possible? From our physical experience, we know that a velocity, $v$, in a direction making an angle, $\phi$, with the horizontal is equivalent in effect to the combined effects of $v_0 \cos \phi$ and $v_0 \sin \phi$ in the $x$- and $y$-directions, respectively (see Fig. 31).

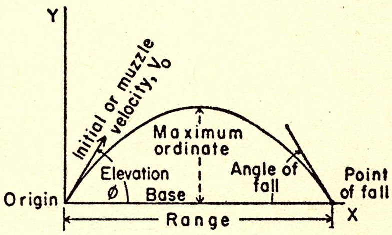



Fig. 31

In solving the equations grouped as Ia, we must make the solutions yield these initial velocities (when $t=0$).

Now, if $\frac{dv_y}{dt} = -32.2$, then $v_y = -32.2\,t + \xi$ satisfies this differential equation.

But since, when $t=0$, $v_y = v_0 \sin \phi$, we have:

$$t=0, \qquad v_y = (-32.2) \times 0 + \xi = v_0 \sin \phi$$
$$\therefore \xi = v_0 \sin \phi$$

and similarly,

$$t=0, \qquad v_x = v_0 \cos \phi$$

These together give as the first stage in the solution:

$$v_y = \frac{dy}{dt} = v_0 \sin \phi - 32.2t$$

$$v_x = \frac{dx}{dt} = v_0 \cos \phi \qquad \textbf{Ib}$$

a new pair of differential equations, where it is understood that when $t=0$, $x=y=0$.

Carrying on, that is seeking equations which satisfy Ib, we find:

$$y = (v_0 \sin \phi)t - \frac{32.2t^2}{2} + \xi_1$$

$$x = (v_0 \cos \phi)t + \xi_2$$

On putting 0 for $t$, we have $0=\xi_1$, $0=\xi_2$, and our path is then given by

$$\begin{cases} y=(v_0 \sin \phi)t-16.1t^2 \\ x=(v_0 \cos \phi)t \end{cases} \qquad \text{II}$$

called the parametric equations of the trajectory.

What does this path look like? What is the range? We can approximate the path by plotting $x$ and $y$ for various values of $t$, or we can eliminate $t$ and get

$$y=\frac{v_0 \sin \phi}{v_0 \cos \phi}x-\frac{16.1x^2}{v_0^2\cos^2\phi}$$

$$=x \tan \phi -\frac{16.1}{v_0^2\cos^2\phi}x^2. \qquad \text{III}$$

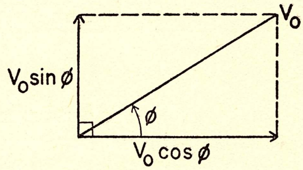


Fig. 32

We agreed that the range was the horizontal distance between the initial point and the point at which the trajectory again met the horizontal axis. At this point, the value of $y$ is again 0; that is, if

$$y=0=x \tan \phi - \frac{16.1}{v_0^2\cos^2 \phi}x^2,$$

there are two values of $x$ at which this is true:

$$x_1=0,$$

and $$x_2=\frac{v_0 \tan \phi \cos^2 \phi}{16.1}$$

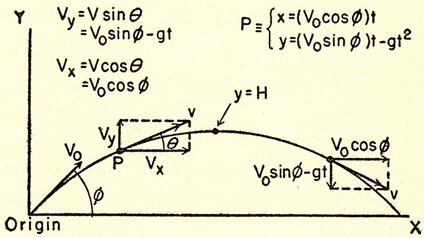


Fig. 33

Since $$16.1=\frac{g}{2},$$

the range is

$$X=x_2-x_1=\frac{v_0^2\dfrac{\sin \phi}{\cos \phi}\cos^2 \phi}{\dfrac{g}{2}}=\frac{v_0^2\, 2 \sin \phi \cos \phi}{g}=\frac{v_0^2 \sin 2 \phi}{g} \qquad \text{IV}$$

Figs. 32 and 33 summarize the trajectory *in vacuo* (in vacuum).

Suppose you had to clear a hill. It would be well to know at least, if the maximum ordinate, $H$, were greater than the height of the hill.

$H$ is a value of $y$ for which $D_ty\equiv\frac{dy}{dt}=0$, in equation II, or for which $D_xy\equiv\frac{dy}{dx}=0$, in equation III. When $\frac{dy}{dt}=0$, $v_0 \sin \phi-gt=0$ or $t=\frac{v_0 \sin \phi}{g}$; whence, for this value of $t$,

$$y=H=\frac{v_0^2 \sin^2\phi}{g}-\frac{gv_0^2 \sin^2 \phi}{2g^2}=\frac{v_0^2 \sin^2\phi}{2g}.$$

For this value of $y$,

$$x=t(v_0 \cos \phi)=\frac{2v_0 \sin \phi}{2g}v_0 \cos \phi$$

$$=\frac{v_0^2 \sin 2\phi}{2g}=\frac{\text{Range}}{2}.$$

Check these results using the cartesian equation, III, for the trajectory.

Now suppose, for elevation $\phi$ and range $X$, the trajectory will not clear an obstruction. Can one use another trajectory for the same range; that is, is there more than one value of $\phi$ for the same initial velocity and range? Let's try: let $X$=some constant, $K$, and find out what you can about $\phi$. Rather directly, take

$$X=\frac{v_0^2 \sin^2 \phi}{g}=k$$

and solve for $\phi$. We get first

$$\sin^2 \phi=\frac{gk}{v_0^2}\leqq 1$$

There are exactly two values of $\phi$ which satisfy this equation, so that $2\phi \leqq 180°$ and $\phi \leqq 90°$. These two values are complementary angles. We have, therefore, for a given range and initial velocity, a choice of two trajectories. Observe that, for the higher trajectory, the total time of flight is the greater.

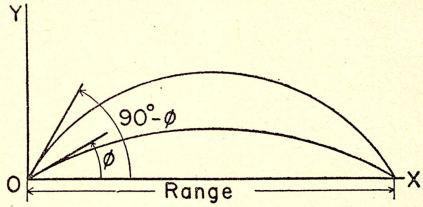



Fig. 34

**TEST YOUR KNOWLEDGE OF THE TRAJECTORY**

**14** Find the parametric equations for the trajectory *in vacuo* for which initial velocity is 1000 feet per second and the $\phi=30°$.

**15** In problem 14, let $\phi=60°$: (a) find parametric equation of the trajectory; (b) find the ranges of the trajectories of problems 14 and 15 (a) respectively; (c) find the maximum ordinates.

**16** If the angle of elevation be held constant, what is the effect of doubling $v_0$ on (a) the range; (b) the maximum ordinate; (c) the time of flight?

**17** Given $v_0=300\sqrt{32}$ feet per second, (a) find equations of trajectories for elevations: 15°, 30°, 45°, 60°, 75°; (b) plot each of these curves on common axes; (c) find ranges in each case; (d) find maximum elevations in each case; (e) find the angle of fall in each case.

**18** Given $\phi=30°$, (a) find equations of trajectories for initial velocities, $v_0=100\sqrt{32}$, $200\sqrt{32}$, $300\sqrt{32}$, and $400\sqrt{32}$ feet per second, respectively; (b) plot these curves on common axes.

## *The trajectory in air*

We cannot overlook the fact, however, that in practice projectiles are fired through the atmosphere. The trajectories traversed will therefore be quite different from the theoretical parabola in the vacuum. A little thought will reveal many difficulties which place the problem beyond the range of our discussion. In a qualitative way, however, it is well to examine some factors involved.

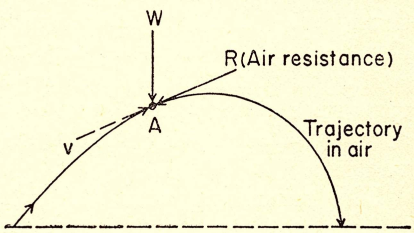



Fig. 35

Let us examine the forces on $A$, as shown in Fig. 35.

The motion is still assumed to be in a vertical plane. There are now two forces acting on the projectile, which, at any instant, is moving along

a tangent to the trajectory, as denoted by velocity vector $v$. At such an instant, $R$ is in the direction of $v$, but in the opposite sense. The effect of $R$ becomes much clearer if we resolve $R$ into two components at right angles, for $R$ produces the same effect as do the two components, $R \cos \theta$, horizontal, and $R \sin \theta$, vertical. Observe the situation as depicted in Fig. 36.

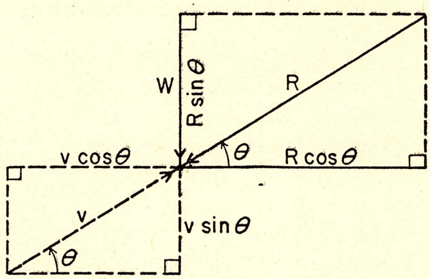


Fig. 36

In the vertical direction, forces $W$ and $R \sin \theta$ are both downward, combining their effects to produce a downward acceleration greater than $g$. Hence, the projectile will not rise so high in air as it can rise in a vacuum with the same initial velocity, $v_0$. Again, in a vacuum there is no resistance to $v_0 \cos \phi$, the initial horizontal velocity. In the atmosphere, however, the force, $R \cos \theta$, produces an acceleration opposite in direction to $v \cos \theta$, slowing up the horizontal flight of the projectile. The range in air, therefore, will not be so great for the same initial velocity as in a vacuum. In fact, the actual range may be only seventy per cent of the theoretical range. On descending in air, the projectile takes a steeper path than on ascending, since the horizontal resistance has been in effect longer. A comparison is shown in Fig. 37.

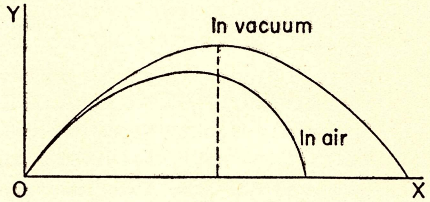


Fig. 37

We have so far scratched the surface of gunnery just enough to see the depth of problem. We cannot here probe any deeper, but we shall examine a practical aspect of gunfire which is guided by astute applications of pure mathematics.

## DISPERSION AND PROBABILITY

Among the many physical factors which determine the ultimate trajectory consider the list in the accompanying table.

**VARIABLES INFLUENCING TRAJECTORY**

1 Quadrant elevation
2 Muzzle (initial) velocity
3 Range
4 Air characteristics (Atmospheric conditions)
   *a* Pressure
   *b* Humidity
   *c* Temperature
   *d* Wind
5 Weight of projectile
6 Shape of projectile
7 Rotation of earth
8 Curvature of earth
9 Trunnion tilt

Any two of the first three variables determine the theoretical parabola obtained as the trajectory on the assumption of a vacuum.

The remaining variables add their influence on the actual trajectory in the atmosphere. When adequate data are obtained on each of these variables, their effects on the point of fall of a projectile can presumably be determined mathematically. We cannot here develop the theory for these actual projectile paths. In fact, only

the results are known to most gunners, in the form of range tables. From these tables, given the data on air characteristics, the muzzle velocity, and the desired range, one can pick out the proper angle of elevation and corrections to firing angle in order to hit the target.

In spite of all these preliminary calculations, it is impossible to govern variations in the wind and other air characteristics. The muzzle velocity is dependent on variations in the powder charge and the projectile. Therefore, in spite of all preliminary efforts to fire accurately under precisely controlled conditions, the points of fall of 100 shots will be ranged about the target in random fashion; that is, a chance element enters. The hits will fill out a rectangle whose length in the direction of fire is, often, about twice its width. More shots will fall close to the center of the rectangle than toward the extremes, and about 99 per cent of the shots will fall inside the rectangle. We refer to this phenomenon as the *dispersion pattern* of the gun.

We shall study the dispersion in the direction of fire only, omitting consideration of deflection from the vertical plane of fire. The quadrant elevation can be determined accurately for a given range. Were the muzzle velocity accurately determined, the points of fall would deviate but slightly from the target whose range was accurately known, but muzzle velocity is dependent on a given powder charge. The rated $v_0$ is determined for a standard weight of charge and a standard temperature at which the charge is kept before firing. Correction can be made for variations in $v_0$ owing to temperature. Corrections can also be made for air density, but in spite of all the corrections that we can make, the *chance variations* shown in the table can take place.

CHANCE VARIATIONS

1 **Quadrant Elevation:** slight deviations from exact value.

2 **Atmospheric conditions:** charges during fire.

3 **Projectile:** slight differences in weight, shaz, location of center of mass, and distance rammed.

4 **Powder charge:** slight variations in weight, temperature, grain size, moisture content, and burning rate.

5 **Erosion:** effect of chemical action of explosion on powder chamber and lining of bore.

The effects of these variations on the range, $X$, of the trajectory for a given quadrant angle, $\phi$, is what is commonly termed a random or chance distribution in the values $X_1, X_2, \ldots, X_n$ of the respective ranges produced by $n$ shots. Experience at the various proving grounds has shown that the distances at which shots from a gun land distribute themselves in the following random fashion: the linear average of these $X_i$, denoted by $\bar{X}$ is the sum of all the $X_i$ divided by $n$; that is,

$$\bar{X}=\frac{X_1+X_2+\ldots+X_n}{n}.$$

This distance is called the *center of impact* or *center of dispersion*

of the ranges, and is, in fact, the *effective range* of the gun, for, in actual experience, when a large number of shots are fired, there are about as many $X_i$ less than $\overline{X}$ as there are $X_i$ greater than $\overline{X}$. We find in experience another important fact about the distribution of the $X_i$—namely, that the distribution can be expressed in terms of the following quantity:

$$r=\frac{2}{3}\sqrt{\frac{(\overline{X}-X_1)^2+(\overline{X}-X_2)^2+\ .\ .\ .\ +(\overline{X}-X_n)^2}{n}}$$

We find this to be the case: let $O$ in Fig. 38 be the center of impact, and stake out four points on either side of $O$ at distances, $r$, $2r$, $3r$, and $4r$, respectively, designating these points as shown. The number of shots falling

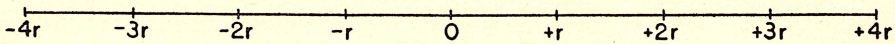


Fig. 38

between $-r$ and $O$ amounts to 25 per cent of the total fired; and the same is true between $O$ and $+r$.

The distribution in the various sectors is illustrated in the following rectangle whose width is assumed to be great enough so that only 1 per cent of the shots will be deflected so widely as to fall outside its vertical sides. Let the central line be at the distance, $\overline{X}$, from the piece. Then stake off lines at distances $\pm r$, $\pm 2r$, $\pm 3r$, and $\pm 4r$ from the piece. The distribution of the shots within the rectangle and with respect to distance from $\overline{X}$ will be as pictured in Fig. 39.

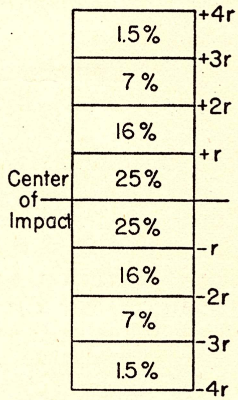


Fig. 39

The dispersion rectangle means this: if a large number of shots are fired from a gun at the quadrant elevation specified for the range, $X$, about 50 per cent will fall within a distance $r$ yards greater than or less than $\overline{X}$; about 16 per cent of the shots will fall in regions indicated, and so on. About 99 per cent of the shots will fall between the upper and lower limits of this rectangle, and about one shot in one hundred will go astray.

If we took the linear average of the actual ranges,

$$\overline{X}=\frac{X_1+X_2+\ .\ .\ .\ +X_n}{n},$$

it would be very close to $X$, the tabular range for which the quadrant elevation was chosen. About as many shots would be greater than $\overline{X}$ or $X$ as would be smaller. Our experience leads us to this conclusion: when speculating on where the shots will land, we say there is a probability of $\frac{1}{2}$ that a shot will land between $r$ and $-r$ of $\overline{X}$.

In actual practice, it is very difficult to know whether the true

distance of the target is in fact $X$; hence, we must seek by trial an $\overline{X}$ and work on the principle that we can adjust the elevation so that this center of impact falls close to the target. We then believe that we have an even chance (1 to 1 odds or $\frac{1}{2}$ probability) of hitting the target if it spreads over $-r$ ro $r$ of our rectangle; and we have 0.82 of a chance of hitting the target if it spreads over $-2r$ to $2r$.

We have used the concept of probability without definition and for a good reason. It is difficult to convey a convincing notion of probability without going into the subject deeply. It is very easy to convey false and misleading notions. All that we are saying is this: if, in making 1,000 trials at producing a definite result, we get the result to occur 400 times; and if there is no reason, so far as we know, why the result should or should not occur, we say that there is a probability of $\frac{4}{10}$ that the desired result will take place. The desired result is what we call a chance occurrence. We cannot say that it will occur at a given trial, but we are confident that, in a great many trials, the result will occur about $\frac{4}{10}$ of the time. Experience in the firing of guns has in the past borne out the conviction that such a notion of probability can be used, not only to determine dispersion patterns in the proving ground, but also to adjust fire so as to place the center of impact close to the center of the target. Experience has provided us with so-called random frequency distributions, and the intuition of some mathematicians has led to methods for making up such frequency distributions. We shall examine frequency distributions a little more closely.

First, we shall pose this situation: there is a rigid flat circular disc which looks red on one side and blue on the other. If it were tossed in the air in no special manner, would you hazard a bet, and at what odds, that the disc will fall red side up? If you were like other people—in particular, if you were a patron of the red and the black—you would place even odds on the red. Intuitively, we believe the chances equal for the red or the blue to land face up. If you were less human and more scientific, you would say, "Let me test this disc", and you would have the disc tossed at "random" (let's not be too philosophic about this word, *random*, and accept the Webster definition) many, many times. When we were all very tired, you would feel that enough trials had been made, and tally the reds and the blues. It might have happened that the red came up 3723 times, while the blue came up 3731 times, and you would proudly say that the probability, when the given disc is tossed, that it will fall red side up is $\frac{3723}{7454}$.

In the first case, we thought as follows: so far as we can tell, it is equally likely that the disc can fall red or blue side up. Hence, since the disc can fall in one of two ways (not on edge), and since one of these ways is red, the probability that the disc will fall red side up is:

$$p=\frac{1}{2}.$$

The other attitude is this: a very great many actual trials indicate the pattern of occurrence; at the 7454th trial, the ratio of the number of favorable trials

to the total was approximately one-half; the probability that the disc will fall red side up is

$$p = \lim_{n \to \infty} \frac{f_n}{n},$$

but since we must stop the trials some time, we shall do so when $\frac{f_n}{n}$ stays fairly constant. At the 7454th trial, we thought this state had been reached. For most practical purposes, the first point of view is effective.

In gunnery, we cannot say that events are equally likely, but we must infer from our experience in firing. On the other hand, we cannot make too many trials but, when we test a gun and find that it passes specifications, we use it on the principle that, when a great number of shots are fired for a given range, these shots will fall within the regions of the rectangle of dispersion approximately in accordance with the normal-frequency distribution.

The interesting fact is that this frequency distribution in gunnery is closely related to results from tossing heads and tails (or reds and blues). Suppose you tossed 4 pennies; how often would 2 heads and 2 tails come up? Let's answer this, assuming that we have not the time for 10,000 trials. Observe that we could get our results in the ways shown in the first column. These are the only ways in which our result can occur; each is different from the other, and one is as likely as the other. The other ways in which the 4 coins can fall are shown in columns two and three:

| | | |
|---|---|---|
| H H T T | H H H T | T T T H |
| H T H T | H H T H | T T H T |
| H T T H | H T H H | T H T T |
| T T H H | T H H H | H T T T |
| T H H T | H H H H | T T T T |
| T H T H | | |

Altogether there are 16 different ways out of which in 6 ways 2 heads and 2 tails can appear. We say that the probability that, when 4 coins are tossed, 2 heads and 2 tails will appear is

$$\frac{6}{16} = \frac{3}{8}.$$

The amusing thing is this: the number of ways in which different combinations of heads and tails can occur—that is, the frequency distribution of these events—is given by the coefficients of the binomial expansion, $(a+b)^4$. If there were 5 coins, $(a+b)^5$ would apply. For 4 coins, we have:

| | | | |
|---|---|---|---|
| 4H | 0T | 1 | $1$ |
| 3H | 1T | 4 | $n$ |
| 2H | 2T | 6 | $\frac{n(n-1)}{2}$ |
| 1H | 3T | 4 | $\frac{n(n-1)}{2 \cdot 3}$ |
| 0H | 4T | 1 | $\frac{n(n-1)(n-2)}{2 \cdot 3 \cdot 4}$ |
| | | 16 different ways | |

Consider the following proposals: let 9 coins be tossed; consider occurrence 5H and 4T to be equivalent to a shot that lies within two units left of the center of impact; while 4H and 5T is equivalent to shots falling within $2u$ to right of center; 6H3T equivalent to a shot between $4u$ and $2u$ left, and 3H6T equivalent to a shot between $2u$ and $4u$ right, and so on. Show that this frequency distribution is like a normal distribution, and in accordance with the rectangle of dispersion in Fig. 39.

The frequency is as given in Fig. 40. Observe that it is symmetrical so that we need study but one-half of the distribution. Let us find out whether there exists an $r$, such that 25 per cent of the shots will fall within $r$ units on one side of the center of impact, where $O$ is approximately at the center of impact because of the symmetry of distribution about $O$.

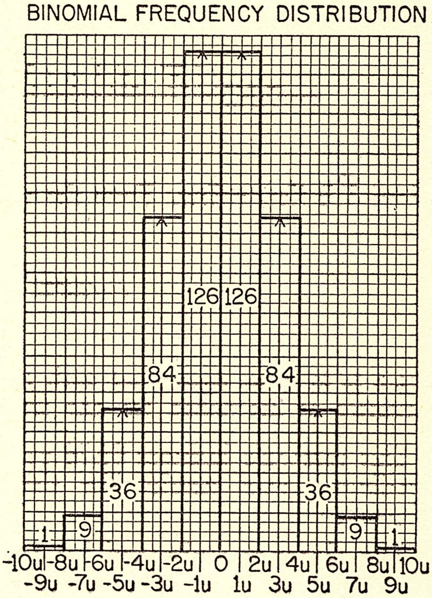


Fig. 40

We shall assume now that $O$ is at $\overline{X}$ and that a shot falling between $-4u$ and $-2u$ will, on the average, fall at $-3u$. Hence, since there are 84 shots in that region, $\overline{X}-X_{127}+\overline{X}-X_{128}+\ \ldots\ +\overline{X}-X_{210}$ is approximately equal to $84(3u)$. In the formula,

$$r=\frac{2}{3}\sqrt{\frac{(\overline{X}-X_1)^2+(\overline{X}-X_2)^2+\ \ldots\ +(\overline{X}-X_n)^2}{n}},$$

all the different $(\overline{X}-X_i)$'s are in this case multiples of $u$. Because of symmetry, we shall study only one-half of the distribution. All the computations are contained in the table below:

| Region | No. of Shots | $\overline{X}-X_i$ | $\lvert\overline{X}-X_i\rvert$ | $f_i\lvert\overline{X}-X_i\rvert$ (Average) | $f_i(\overline{X}-X_i)^2$ |
|---|---|---|---|---|---|
| 0 to $2u$ | 126 | $-u$ | $+u$ | $126u$ | $126u^2$ |
| $2u$ to $4u$ | 84 | $-3u$ | $+3u$ | $252u$ | $756u^2$ |
| $4u$ to $6u$ | 36 | $-5u$ | $+5u$ | $180u$ | $900u^2$ |
| $6u$ to $8u$ | 9 | $-7u$ | $+7u$ | $63u$ | $441u^2$ |
| $8u$ to $10u$ | 1 | $-9u$ | $+9u$ | $9u$ | $81u^2$ |
| Sum | 256 | | | $630u$ | $2304u^2$ |

$$r=\frac{2}{3}\sqrt{\frac{2304u^2}{256}}=\frac{2}{3}\sqrt{9u^2}=2u.$$

Hence, if the above frequency distribution is normal or nearly normal, we ought to find about 25 per cent of our shots in the region, 0 to $2u$. There are $2\times256=512$ shots altogether, 25 per cent of which is 128. This checks very close so far.

In the field, however, computing with squares of $(\overline{X}-X_i)$ may be too much.

Fortunately, our center of impact is close to our *median* (a point so chosen in a series that half of the cases in the series are greater and half are smaller). Using the median, we can obtain

$$r=0.845\frac{(|\overline{X}-X_1|+|\overline{X}-X_2|+\ldots+|\overline{X}-X_n|)}{n}$$

where $\overline{X}$ is the median. From the above table,

$$r=0.845\frac{(630u)}{256}=2.08u,$$

which is in close agreement with the value just found. Fig. 41 shows the rectangle of dispersion for the frequency distribution based on $(a+b)^9$.

This rectangle includes 98.76 per cent of the shots distributed very nearly like the so-called normal frequency.

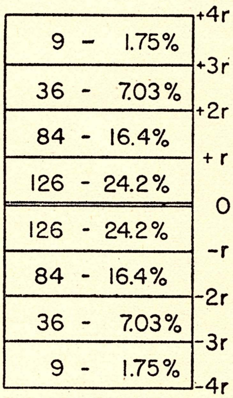


Fig. 41

The distance embraced by $4r$ is called a *fork*. With a given type of gun, firing tables are supplied which specify the number of mils of change in quadrant elevation in order to increase the range by the number of yards embraced by one fork. This is, of course, variable with the range. Tables in this issue of PRACTICAL MATHEMATICS provide data typical of that included in firing tables for a seventy-five millimeter field piece.

In actual practice, an observer guides the gunner by *spotting* (determining the position of a burst (shot) relative to the target). From this information and from data in the firing tables, the gunner can modify the quadrant angle until the center of impact is on or close to the target.

#### TEST YOUR ABILITY TO COMPUTE PERCENTAGE DISTRIBUTION

**19** Following the method described on pages 757 and 758, determine $r$ by the two methods there used: assume that 2048 shots were fired (sum of coefficients in $(a+b)^{11}$); assume further that the points of fall are distributed in the manner described above, but with six regions, each $2u$ units wide, lying on either side of the center of impact; find the percentage distribution of shots falling into the regions 0 to $r$, $r$ to $2r$, $2r$ to $3r$, and $3r$ to $4r$, either side of the center of impact. Draw the rectangle of dispersion. Follow table on page 757.

# *Solutions to Problems and Exercises in Issue 11*

## *THE MATHEMATICS OF CHEMISTRY*

### MOLECULAR WEIGHTS AND PERCENTAGE COMPOSITION

**1** 286.164
**2** 84.015
**3** 685.63
**4** 859.274
**5** 775.666
**6** 46.068
**7** Mg 41.69%; O 54.85%; H 3.45%
**8** Ca 50.03%; C 14.99%; N 34.97%
**9** C 31.99%; H 6.71%; N 18.66%; O 42.63%
**10** Na 14.34%; Sb 25.31%; S 26.65%; H 3.77%; O 29.93%
**11** Na 18.53%; S 25.83%; O 51.57%; H 4.07%
**12** C 74.87%; H 25.13%
**13** 7.9594 g. Al
**14** 97.454 g. Mg
**15** Hg 92.6%; O 7.4%
**16** (a) 139.61 g. (b) 860.39 g.

### VALENCIES

**17** −3
**18** +5
**19** −2
**20** +3
**21** $-\frac{1}{3}$
**22** N = −3, H = +1, S = +7, O = −2
**23** Na = +1, S = +2, O = −2
**24** K = +1, P = +5, O = −2
**25** Na = +1, Mn = +7, O = −2
**26** Li = +1, Mn = +6, O = −2
**27** −1
**28** +3
**29** +1
**30** +5
**31** +7
**32** Valence $SO_4 = -2$
**33** Valence $NH_4 = +1$
**34** Valence $PO_4 = -3$
**35** Valence $OH = -1$
**36** Valence $NO_3 = -1$

### BALANCING EQUATIONS

**37** 1+1 → 1+2+2
**38** 1+12+23 → 2+21+1+12
**39** 1+4+6 → 2+4+2+3
**40** 1+3+1 → 1+3+2
**41** 2+1 → 1+1
**42** 1+3+3 → 1+3
**43** 1+4+10 → 1+2+4+4
**44** 2+3+2 → 1+5+2
**45** 2+3 → 2+3+3
**46** 3+28+4 → 6+9+28

### DETERMINING WEIGHTS

**47** 20.988 g.
**48** (a) 931.133 g.
(b) 102.670 g.
(c) 133.788 g.
**49** 2,484.03 g.
**50** 3.86172 g.
**51** 1.875 g.
**52** 0.1751 g.
**53** 4.6 ppM

### GASES AND VAPORS

**54** 31.33185 g.
**55** 3 liters
**56** 3 mg. As; 0.86 c.c.
**57** CO in excess by 12.5 c.c.
12.5 c.c. $CO_2$ formed
**58** 98.9184 Calculated weight: 98.924
**59** 58.016 Calculated weight: 58.12
**60** 7.644 l.
**61** 730 c.c.
**62** 33.67%
**63** 41.975 liters
**64** (a) 8,859,000 l. (b) 1,772,000 l.

### SPECIFIC GRAVITY

**65** 159.25 g.
**66** 306.675 g.
**67** 400.197 g.
**68** 23.458%
**69** 17.85 c.c.
**70** 418.12515 g.
**71** 5.605 l.
**72** 1.228%

### MOLARITY

**73** 374.565 g.
**74** 0.319M
**75** 1.66M
**76** $4 \times 10^{-5}$M
**77** 45.2752 g.
**78** (a) 1.009 l. glycol
(b) 1.125 Kg.

### FORMING SOLUTIONS

**79** M.W. 112.32; E.W. 56.17
**80** M.W. 42.397; E.W. same
**81** M.W. 150.83; E.W. 50.277
**82** M.W. 366.03; E.W. 61.005
**83** M.W. 379.346; E.W. 189.673
**84** M.W. 859.274; E.W. 71.60617
**85** 18.9105 g.
**86** 0.012N
**87** 8.36 c.c.
**88** 39.91 c.c.
**89** 3.68 c.c.
**90** 3.38 *pH*
**91** 5*pH*

## *THE MATHEMATICS OF HEAT*

### CONVERTING SCALE READINGS

**1** 86° F.
**2** 23° F.
**3** 104° F.
**4** −26.1° C.
**5** 82.2° C.
**6** 35° C.
**7** 167° F.
**8** 28° R.
**9** 212° F., 32° F.
**10** 0°
**11** 58° F.
**12** 37° C.

### EXPANSION OF SOLIDS

**13** 0.114″
**14** 0.601″
**15** 100.081′
**16** 568° C.

### CUBICAL EXPANSION

**17** 0.50 cu. in.
**18** 1.95 cu. in. (Coeff. = $30 \times 10^{-6}$)
**19** 0.415 sq. in.

### EXPANSION OF LIQUIDS

**20** 231 in.
**21** 88.6 in.
**22** −0.000003 at 2° C.
−0.000007 at 0° C. (32° F.)
0.0006 at 84° C.

### EXPANSION OF GASES

**23** 631 c.c.
**24** 181,500 cu. ft.
**25** 80° C.
**26** 697 c.c.
**27** 399 c.c.
**28** 767 c.c.
**29** 36.2 cm.

### TEMPERATURES OF MIXTURES

**30** 26,250 calories
**31** 16,000 calories
**32** 244,944 calories
**33** 0.0912:1
**34** 19.7° C.

### CALORIMETRY

**35** 545.6 calories
**36** 20.8 lb. ice

# *Tables and Formulas*

## TABLE LXIX

### *WIRE SIZES FOR DISTANCE VOLTAGE DROP*

| Capacity Amperes | Distribution in Feet to Center of Distribution | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **20** | **30** | **40** | **50** | **60** | **70** | **80** | **90** | **100** | **120** | **140** | **160** | **180** | **200** | **240** | **280** | **320** | **360** |
| **1** | .. | .. | .. | .. | .. | .. | .. | .. | .. | .. | ... | .... | .... | .... | 16 | 15 | 15 | 14 |
| **1.5** | .. | .. | .. | .. | .. | .. | .. | .. | .. | .. | ... | 16 | 15 | 15 | 14 | 14 | 13 | 12 |
| **2** | .. | .. | .. | .. | .. | .. | .. | .. | .. | 16 | 15 | 15 | 14 | 14 | 13 | 12 | 12 | 11 |
| **3** | .. | .. | .. | .. | .. | .. | 16 | 15 | 15 | 14 | 14 | 13 | 12 | 12 | 11 | 11 | 10 | 9 |
| **4** | .. | .. | .. | .. | 16 | 15 | 15 | 14 | 14 | 13 | 12 | 12 | 11 | 11 | 10 | 9 | 9 | 8 |
| **5** | .. | .. | .. | 16 | 15 | 14 | 14 | 13 | 13 | 12 | 11 | 11 | 10 | 10 | 9 | 8 | 8 | 7 |
| **6** | .. | .. | 16 | 15 | 14 | 14 | 13 | 12 | 12 | 11 | 11 | 10 | 9 | 9 | 8 | 8 | 7 | 7 |
| **7** | .. | 16 | 15 | 14 | 14 | 13 | 12 | 12 | 11 | 11 | 10 | 9 | 9 | 8 | 7 | 7 | 6 | 6 |
| **8** | .. | 16 | 15 | 14 | 13 | 12 | 12 | 11 | 11 | 10 | 9 | 9 | 8 | 8 | 7 | 7 | 6 | 5 |
| **9** | .. | 15 | 14 | 13 | 12 | 12 | 11 | 11 | 10 | 9 | 9 | 8 | 8 | 7 | 7 | 6 | 5 | 5 |
| **10** | 16 | 15 | 14 | 13 | 12 | 11 | 11 | 10 | 10 | 9 | 8 | 8 | 7 | 7 | 6 | 5 | 5 | 4 |
| **12** | 16 | 14 | 13 | 12 | 11 | 11 | 10 | 9 | 9 | 8 | 8 | 7 | 7 | 6 | 5 | 5 | 4 | 4 |
| **14** | 15 | 14 | 12 | 11 | 11 | 10 | 9 | 9 | 8 | 7 | 7 | 6 | 6 | 5 | 5 | 4 | 3 | 3 |
| **16** | 15 | 13 | 12 | 11 | 10 | 9 | 9 | 8 | 8 | 7 | 7 | 6 | 5 | 5 | 4 | 3 | 3 | 2 |
| **18** | 14 | 12 | 11 | 10 | 9 | 9 | 8 | 8 | 7 | 7 | 6 | 5 | 5 | 4 | 4 | 3 | 2 | 2 |
| **20** | 14 | 12 | 11 | 10 | 9 | 8 | 8 | 7 | 7 | 6 | 5 | 5 | 4 | 4 | 3 | 2 | 2 | 1 |
| **25** | 13 | 11 | 10 | 9 | 8 | 7 | 7 | 6 | 6 | 5 | 4 | 4 | 3 | 3 | 2 | 1 | 1 | 0 |
| **30** | 12 | 10 | 9 | 8 | 7 | 7 | 6 | 6 | 5 | 4 | 4 | 3 | 3 | 2 | 1 | 1 | 0 | 0 |
| **35** | 11 | 10 | 8 | 7 | 7 | 6 | 5 | 5 | 4 | 4 | 3 | 2 | 2 | 1 | 1 | 0 | 00 | 00 |
| **40** | 11 | 9 | 8 | 7 | 6 | 5 | 5 | 4 | 4 | 3 | 2 | 2 | 1 | 1 | 0 | 00 | 00 | 000 |
| **45** | 10 | 9 | 7 | 6 | 6 | 5 | 4 | 4 | 3 | 3 | 2 | 1 | 1 | 0 | 00 | 00 | 000 | 000 |
| **50** | 10 | 8 | 7 | 6 | 5 | 4 | 4 | 3 | 3 | 2 | 1 | 1 | 0 | 0 | 00 | 000 | 000 | 0000 |
| **60** | 9 | 7 | 6 | 5 | 4 | 4 | 3 | 3 | 2 | 1 | 1 | 0 | 0 | 00 | 000 | 000 | 0000 | 0000 |
| **70** | 8 | 7 | 5 | 4 | 4 | 3 | 2 | 2 | 1 | 1 | 0 | 00 | 00 | 000 | 000 | 0000 | 0000 | .... |
| **80** | 8 | 6 | 5 | 4 | 3 | 2 | 2 | 1 | 1 | 0 | 00 | 00 | 000 | 000 | 0000 | 0000 | .... | .... |
| **90** | 7 | 6 | 4 | 3 | 3 | 2 | 1 | 1 | 0 | 00 | 00 | 000 | 000 | 0000 | 0000 | .... | .... | .... |
| **100** | 7 | 5 | 4 | 3 | 2 | 1 | 1 | 0 | 0 | 00 | 000 | 000 | 0000 | 0000 | .... | .... | .... | .... |
| **120** | 6 | 4 | 3 | 2 | 1 | 1 | 0 | 0 | 00 | 00 | 000 | 0000 | 0000 | .... | .... | .... | .... | .... |

Wire Size, A.W.G., for 2% loss on 110 volts (for 220 volts, divide by 2.)

## TABLE LXX

### RELATION OF LOAD, DISTANCE, LOSS, AND CONDUCTOR SIZE OF 2-WIRE CIRCUITS

| Wire Size, A.W.G. 110-volt circuit | Wire Size, A.W.G. 220-volt circuit | Line Loss in Percentage of Rated Voltage. Power Loss in Percentage of Delivered Power: 1 | 1.5 | 2 | 3 | 4 | 5 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0000 | 21,550 | 32,325 | 43,100 | 64,650 | 86,200 | 107,750 | 129,300 | 172,400 | 215,500 |
| | 000 | 17,080 | 25,620 | 34,160 | 51,240 | 68,320 | 85,400 | 102,480 | 136,640 | 170,800 |
| | 00 | 13,550 | 20,325 | 27,100 | 40,650 | 54,200 | 67,750 | 81,300 | 108,400 | 135,500 |
| 0000 | 0 | 10,750 | 16,125 | 21,500 | 32,250 | 43,000 | 53,750 | 64,500 | 86,000 | 107,500 |
| 000 | 1 | 8,520 | 12,780 | 17,040 | 25,560 | 34,080 | 42,600 | 51,120 | 68,160 | 85,200 |
| 00 | 2 | 6,750 | 10,140 | 13,520 | 20,280 | 27,040 | 33,800 | 40,560 | 54,080 | 67,600 |
| 0 | 3 | 5,360 | 8,040 | 10,720 | 16,080 | 21,440 | 26,800 | 32,160 | 42,880 | 53,600 |
| 1 | 4 | 4,250 | 6,375 | 8,500 | 12,750 | 17,000 | 21,250 | 25,500 | 34,000 | 42,500 |
| 2 | 5 | 3,370 | 5,055 | 6,740 | 10,110 | 13,480 | 16,850 | 20,220 | 26,960 | 33,700 |
| 3 | 6 | 2,670 | 4,005 | 5,340 | 8,010 | 10,680 | 13,350 | 16,020 | 21,360 | 26,700 |
| 4 | 7 | 2,120 | 3,180 | 4,240 | 6,360 | 8,480 | 10,600 | 12,720 | 16,960 | 21,200 |
| 5 | 8 | 1,680 | 2,520 | 3,360 | 5,040 | 6,720 | 8,400 | 10,800 | 13,440 | 16,800 |
| 6 | 9 | 1,330 | 1,995 | 2,660 | 3,990 | 5,320 | 6,650 | 7,980 | 10,640 | 13,300 |
| 7 | 10 | 1,055 | 1,582 | 2,110 | 3,165 | 4,220 | 5,275 | 6,330 | 8,440 | 10,550 |
| 8 | 11 | 838 | 1,257 | 1,675 | 2,514 | 3,350 | 4,190 | 5,028 | 6,700 | 8,380 |
| 9 | 12 | 665 | 997 | 1,330 | 1,995 | 2,660 | 3,320 | 3,990 | 5,320 | 6,650 |
| 10 | 13 | 527 | 790 | 1,054 | 1,580 | 2,108 | 2,635 | 3,160 | 4,215 | 5,270 |
| 11 | 14 | 418 | 627 | 836 | 1,254 | 1,672 | 2,090 | 2,508 | 3,344 | 4,180 |
| 12 | .... | 332 | 498 | 665 | 997 | 1,330 | 1,660 | 1,995 | 2,660 | 3,325 |
| 14 | .... | 209 | 313 | 418 | 627 | 836 | 1,045 | 1,354 | 1,672 | 2,090 |

Ampere-feet = (Amperes × length of one wire)

## TABLE LXXI

## *DIMENSIONS AND RESISTANCE OF COPPER WIRE AND CABLE*

AMERICAN WIRE GAGE (A.W.G.)

| Gage No. A.W.G. | Diam. *Mils* | Cross Section *Cir. mils* | Cross Section *Sq. in.* | Ohms per 1000 Ft. at 25° C. (77° F.) | Lb. per 1000 Ft. |
|---|---|---|---|---|---|
| **0000** | 460 | 212,000 | 0.166 | 0.0500 | 641 |
| **000** | 410 | 168,000 | .132 | .0630 | 508 |
| **00** | 365 | 133,000 | .105 | .0795 | 403 |
| **0** | 325 | 106,000 | .0829 | .100 | 319 |
| **1** | 289 | 83,700 | .0657 | .126 | 253 |
| **2** | 258 | 66,400 | .0521 | .159 | 201 |
| **3** | 229 | 52,600 | .0413 | .201 | 159 |
| **4** | 204 | 41,700 | .0328 | .253 | 126 |
| **5** | 182 | 33,1 0 | .0260 | .320 | 100 |
| **6** | 162 | 26,300 | .0206 | .403 | 79.5 |
| **7** | 144 | 20,800 | .0164 | .508 | 63.0 |
| **8** | 128 | 16,500 | .0130 | .641 | 50.0 |
| **9** | 114 | 13,100 | .0103 | .808 | 39.6 |
| **10** | 102 | 10,400 | .00815 | 1.02 | 31.4 |
| **11** | 91 | 8,230 | .00647 | 1.28 | 24.9 |
| **12** | 81 | 6,530 | .00513 | 1.62 | 19.8 |
| **13** | 72 | 5,180 | .00407 | 2.04 | 15.7 |
| **14** | 64 | 4,110 | .00323 | 2.58 | 12.4 |
| **15** | 57 | 3,260 | .00256 | 3.25 | 9.86 |
| **16** | 51 | 2,580 | .00203 | 4.09 | 7.82 |
| **17** | 45 | 2,050 | .00161 | 5.16 | 6.20 |
| **18** | 40 | 1,620 | .00128 | 6.51 | 4.92 |
| **19** | 36 | 1,290 | .00101 | 8.21 | 3.90 |
| **20** | 32 | 1,020 | .000802 | 10.4 | 3.09 |
| **21** | 28.5 | 810 | .000636 | 13.1 | 2.45 |
| **22** | 25.3 | 642 | .000505 | 16.5 | 1.94 |
| **23** | 22.6 | 509 | .000400 | 20.8 | 1.54 |
| **24** | 20.1 | 404 | .000317 | 26.2 | 1.22 |
| **25** | 17.9 | 320 | .000252 | 33.0 | 0.970 |
| **26** | 15.9 | 254 | .000200 | 41.6 | .769 |
| **27** | 14.2 | 202 | .000158 | 52.5 | .610 |
| **28** | 12.6 | 160 | .000126 | 66.2 | .484 |
| **29** | 11.3 | 127 | .0000995 | 83.5 | .384 |
| **30** | 10.0 | 101.0 | .0000789 | 105 | .304 |
| **31** | 8.9 | 79.7 | .0000626 | 133 | .241 |
| **32** | 8.0 | 63.2 | .0000496 | 167 | .191 |
| **33** | 7.1 | 50.1 | .0000394 | 211 | .152 |
| **34** | 6.3 | 39.8 | .0000312 | 266 | .120 |
| **35** | 5.6 | 31.5 | .0000248 | 336 | .0954 |
| **36** | 5.0 | 25.0 | .0000196 | 423 | .0757 |
| **37** | 4.5 | 19.8 | .0000156 | 533 | .0600 |
| **38** | 4.0 | 15.7 | .0000123 | 673 | .0476 |
| **39** | 3.5 | 12.5 | .0000098 | 848 | .0377 |
| **40** | 3.1 | 9.9 | .0000078 | 1070 | .0299 |

Resistivity of pure copper at 20° C. = 0.15328 ohm per meter.

(U. S. Bureau of Standards)

## TABLE LXXII

### CORRECTED DENSITY AND TEMPERATURE

| Height of Battery With Reference to the MDP (feet) | Change in— Density (Per Cent) | Change in— Temperature (°F.) |
|---|---|---|
| +600 | −1.8 | −1.2 |
| +500 | −1.5 | −1.0 |
| +400 | −1.2 | −0.8 |
| +300 | −0.9 | −0.6 |
| +200 | −0.6 | −0.4 |
| +100 | −0.3 | −0.2 |
| Same | 0 | 0 |
| −100 | +0.3 | +0.2 |
| −200 | +0.6 | +0.4 |
| −300 | +0.9 | +0.6 |
| −400 | +1.2 | +0.8 |
| −500 | +1.5 | +1.0 |
| −600 | +1.8 | +1.2 |

Density decreases 0.3 per cent for each 100 feet battery is above the MDP.
Temperature decreases 0.2° F. for each 100 feet battery is above the MDP.
(Source: *Abbreviated Firing Tables*, U. S. War Department Technical Manual, No. 6-215. Page 17.)

## TABLE LXXIII

### CONVERSION OF DEGREES AND MINUTES TO MILS

| Degrees by 10's | Mils | Degrees | Mils | Minutes by 10's | Mils | Minutes | Mils | Minutes in Tenths | Mils |
|---|---|---|---|---|---|---|---|---|---|
| 10° | 177.78 | 1° | 17.78 | 10′ | 2.96 | 1′ | 0.30 | 0.1′ | 0.03 |
| 20° | 355.56 | 2° | 35.56 | 20′ | 5.93 | 2′ | 0.59 | 0.2′ | 0.06 |
| 30° | 533.33 | 3° | 53.33 | 30′ | 8.89 | 3′ | 0.89 | 0.3′ | 0.09 |
| 40° | 711.11 | 4° | 71.11 | 40′ | 11.85 | 4′ | 1.19 | 0.4′ | 0.12 |
| 50° | 888.89 | 5° | 88.89 | 50′ | 14.82 | 5′ | 1.48 | 0.5′ | 0.15 |
| 60° | 1066.67 | 6° | 106.67 | | | 6′ | 1.78 | 0.6′ | 0.18 |
| 70° | 1244.44 | 7° | 124.44 | | | 7′ | 2.07 | 0.7′ | 0.21 |
| 80° | 1422.22 | 8° | 142.22 | | | 8′ | 2.37 | 0.8′ | 0.24 |
| 90° | 1600.00 | 9° | 160.00 | | | 9′ | 2.67 | 0.9′ | 0.27 |
| 100° | 1777.78 | | | | | | | | |
| 110° | 1955.56 | | | | | | | | |
| 120° | 2133.33 | | | | | | | | |
| 130° | 2311.11 | | | | | | | | |
| 140° | 2488.89 | | | | | | | | |
| 150° | 2666.67 | | | | | | | | |
| 160° | 2844.44 | | | | | | | | |
| 170° | 3022.22 | | | | | | | | |

**Conversion Factors**

| | | |
|---|---|---|
| 1 yard = 0.9144 meters | −1° = 17.7778 mils | −1 mil = 0.0563° |
| 1 meter = 1.0936 yards | 1′ = 0.2963 mils | 1 mil = 3.375′ |

## TABLE LXXIV

## *FIRING TABLES, 75-MM GUN*

Characteristics of 75-mm gun M1897, M1897A1, M1897A2, M1897A3, and M1897A4, firing HE shell Mk. I and shrapnel Mk. I:

### 75-mm Gun

| | | |
|---|---|---|
| Diameter of the bore between lands | 2.953 | in. |
| Diameter of the bore between grooves | 2.992 | in. |
| Total length | 107.126 | in. |
| Length of rifled portion | 87.772 | in. |
| Travel of projectile | 89.9 | in. |
| Capacity of powder chamber | 83.0 | cu. in. |
| Number of grooves | 24.0 | cu. in. |
| Character of rifling ... uniform twist 1 in | 25.6 | calibers |
| Maximum pressure for which gun is designed | 36,000 | lb./sq. in. |
| Weight of gun and breech mechanism | 1,035 | lb. |

### 75-mm Gun Carriage M2

| | On Wheels (*mils*) | On Firing Jack (*mils*) |
|---|---|---|
| Maximum traverse, right | 800 | 800 |
| Maximum traverse, left | 711 | 711 |
| Least possible elevation | −178 | −178 |
| Greatest possible elevation | 818 | 821 |
| Traverse for one turn of traversing handwheel | 19.0 | 19.0 |
| Change in elevation for one turn of elevating handwheel | 10 | 10 |
| | *Yards* | *Yards* |
| Maximum range scale setting | 9,760 | 9,760 |

### 75-mm Gun Carriage M1897 (and Modifications)

| | | |
|---|---|---|
| Total traverse (one-half on each side) | 106 | mils |
| Least possible elevation | −178 | mils |
| Greatest possible elevation | 338 | mils |
| Traverse for one turn of traversing handwheel | 1.8 | mils |
| Change in elevation for one turn of elevating handwheel | 8 | mils |
| Maximum range scale setting | 5,500 | meters |

### Projectile-Mean Weight of Fuzed Projetile in Pounds

*Shrapnel.*—Standardized as 15.96 pounds.
*HE shell Mk. I.*—P. D. fuzes M46 and M47.
Variations in weight are indicated by markings stenciled on the projectile as follows:

| Marking | Weight |
|---|---|
| L | 11.58 |
| + | 11.91 |
| + + (standard) | 12.24 |
| + + + | 12.57 |
| + + + + | 12.90 |

### Fuzes

21-second combination time and percussion.
Point detonating fuzes:
M46 (nose painted white).....superquick.
M47 (nose painted black).....delay.

(Source: *Abbreviated Firing Tables,* U. S. War Department Technical Manual, No. 6-215. Page 18.)

## TABLE LXXV

### WIND COMPONENTS FOR 1-MILE-PER-HOUR WIND

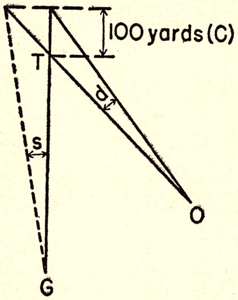
| Chart Direction of Wind | Cross Wind mph | Range Wind mph | Chart Direction of Wind | Cross Wind mph | Range Wind mph |
|---|---|---|---|---|---|
| **0** | 0 | −1.00 | **3200** | 0 | +1.00 |
| **100** | L .10 | − .99 | **3300** | R .10 | + .99 |
| **200** | L .20 | − .98 | **3400** | R .20 | + .98 |
| **300** | L .29 | − .96 | **3500** | R .29 | + .96 |
| **400** | L .38 | − .92 | **3600** | R .38 | + .92 |
| **500** | L .47 | − .88 | **3700** | R .47 | + .88 |
| **600** | L .56 | − .83 | **3800** | R .56 | + .83 |
| **700** | L .63 | − .77 | **3900** | R .63 | + .77 |
| **800** | L .71 | − .71 | **4000** | R .71 | + .71 |
| **900** | L .77 | − .63 | **4100** | R .77 | + .63 |
| **1000** | L .83 | − .56 | **4200** | R .83 | + .56 |
| **1100** | L .88 | − .47 | **4300** | R .88 | + .47 |
| **1200** | L .92 | − .38 | **4400** | R .92 | + .38 |
| **1300** | L .96 | − .29 | **4500** | R .96 | + .29 |
| **1400** | L .98 | − .20 | **4600** | R .98 | + .20 |
| **1500** | L .99 | − .10 | **4700** | R .99 | + .10 |
| **1600** | **L 1.00** | **.00** | **4800** | **R 1.00** | **.00** |
| **1700** | L .99 | + .10 | **4900** | R .99 | − .10 |
| **1800** | L .98 | + .20 | **5000** | R .98 | − .20 |
| **1900** | L .96 | + .29 | **5100** | R .96 | − .29 |
| **2000** | L .92 | + .38 | **5200** | R .92 | − .38 |
| **2100** | L .88 | + .47 | **5300** | R .88 | − .47 |
| **2200** | L .83 | + .56 | **5400** | R .83 | − .56 |
| **2300** | L .77 | + .63 | **5500** | R .77 | − .63 |
| **2400** | L .71 | + .71 | **5600** | R .71 | − .71 |
| **2500** | L .63 | + .77 | **5700** | R .63 | − .77 |
| **2600** | L .56 | + .83 | **5800** | R .56 | − .83 |
| **2700** | L .47 | + .88 | **5900** | R .47 | − .88 |
| **2800** | L .38 | + .92 | **6000** | R .38 | − .92 |
| **2900** | L .29 | + .96 | **6100** | R .29 | − .96 |
| **3000** | L .20 | + .98 | **6200** | R .20 | − .98 |
| **3100** | L .10 | + .99 | **6300** | R .10 | − .99 |
| **3200** | 0 | +1.00 | **6400** | 0 | −1.00 |

This table divides a wind of 1 mile per hour, blowing from the chart direction, into two components: the cross wind, perpendicular to the plane of fire; and the range wind, parallel to the plane of fire. The chart direction is the *Y*-azimuth of the wind direction as given in the metro message (increased by 6400 when necessary) minus the *Y*-azimuth of the direction of fire.

(Source: *Abbreviated Firing Tables*, U. S. War Department Technical Manual, No. 6-215. Page 18.)

# TABLE LXXVI

## *s* AND *d* TABLES




### EXPLANATION

*G* is the gun.

*O* is the observer.

*T* is the target. It is also the angle *T* (*OTG*) (often referred to as the observer displacement and as the target offset in mils).

*R* is range *GT* in thousands of yards.

*r* is the distance *OT* in thousands of yards.

*d* is the deviation, as seen from *O*, caused by a range change of 100 yards (an elevation change of one *c*). Its value depends on the values of *T* and *r*.

*s* is the shift in deflection necessary to keep a shot on the *OT* line when making a range change of 100 yards (an elevation change of one *c*).

### SHIFT IN DEFLECTION

#### *s* Table

| RANGE *GT* IN YARDS | *T* IN MILS | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | **100** | **200** | **300** | **400** | **500** | **600** | **700** | **800** | **900** | **1000** | **1100** | **1150** | **1200** | **1250** | **1300** |
| **2000** | 5 | 10 | 15 | 21 | 27 | 34 | 42 | 51 | 62 | 76 | 95 | 108 | 123 | 142 | 168 |
| **2100** | 5 | 10 | 15 | 20 | 26 | 32 | 40 | 49 | 59 | 73 | 91 | 103 | 117 | 136 | 160 |
| **2200** | 5 | 9 | 14 | 19 | 25 | 31 | 38 | 46 | 56 | 69 | 87 | 98 | 112 | 129 | 153 |
| **2300** | 4 | 9 | 13 | 18 | 24 | 30 | 36 | 44 | 54 | 66 | 83 | 94 | 107 | 124 | 146 |
| **2400** | 4 | 8 | 13 | 18 | 23 | 28 | 35 | 42 | 52 | 64 | 79 | 90 | 102 | 119 | 140 |
| **2500** | 4 | 8 | 12 | 17 | 22 | 27 | 33 | 41 | 50 | 61 | 76 | 86 | 98 | 114 | 134 |
| **2600** | 4 | 8 | 12 | 16 | 21 | 26 | 32 | 39 | 48 | 59 | 73 | 83 | 95 | 109 | 129 |
| **2700** | 4 | 8 | 11 | 16 | 20 | 25 | 31 | 38 | 46 | 56 | 71 | 80 | 91 | 105 | 124 |
| **2800** | 4 | 7 | 11 | 15 | 19 | 24 | 30 | 36 | 44 | 54 | 68 | 77 | 88 | 102 | 120 |
| **2900** | 3 | 7 | 11 | 15 | 19 | 23 | 29 | 35 | 43 | 53 | 66 | 74 | 85 | 98 | 116 |
| **3000** | 3 | 7 | 10 | 14 | 18 | 23 | 28 | 34 | 41 | 51 | 64 | 72 | 82 | 95 | 112 |
| **3200** | 3 | 6 | 10 | 13 | 17 | 21 | 26 | 32 | 39 | 48 | 60 | 67 | 77 | 89 | 105 |
| **3400** | 3 | 6 | 9 | 12 | 16 | 20 | 25 | 30 | 37 | 45 | 56 | 63 | 72 | 84 | 99 |
| **3600** | 3 | 6 | 9 | 12 | 15 | 19 | 23 | 28 | 34 | 42 | 53 | 60 | 68 | 79 | 93 |
| **3800** | 3 | 5 | 8 | 11 | 14 | 18 | 22 | 27 | 33 | 40 | 50 | 57 | 65 | 75 | 88 |
| **4000** | 3 | 5 | 8 | 11 | 14 | 17 | 21 | 25 | 31 | 38 | 48 | 54 | 61 | 71 | 84 |
| **4500** | 2 | 5 | 7 | 9 | 12 | 15 | 19 | 23 | 28 | 34 | 42 | 48 | 55 | 63 | 75 |
| **5000** | 2 | 4 | 6 | 8 | 11 | 14 | 17 | 20 | 25 | 31 | 38 | 43 | 49 | 57 | 67 |
| **5500** | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 19 | 23 | 28 | 35 | 39 | 45 | 52 | 61 |
| **6000** | 2 | 3 | 5 | 7 | 9 | 11 | 14 | 17 | 21 | 25 | 32 | 36 | 41 | 47 | 56 |
| **6500** | 2 | 3 | 5 | 6 | 8 | 10 | 13 | 16 | 19 | 23 | 29 | 33 | 38 | 44 | 52 |
| **7000** | 1 | 3 | 4 | 6 | 8 | 10 | 12 | 15 | 18 | 22 | 27 | 31 | 35 | 41 | 48 |
| **7500** | 1 | 3 | 4 | 6 | 7 | 9 | 11 | 14 | 17 | 20 | 25 | 29 | 33 | 38 | 45 |
| **8000** | 1 | 3 | 4 | 5 | 7 | 9 | 10 | 13 | 16 | 19 | 24 | 27 | 31 | 36 | 42 |
| **8500** | 1 | 2 | 4 | 5 | 6 | 8 | 10 | 12 | 15 | 18 | 22 | 25 | 29 | 33 | 40 |
| **9000** | 1 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 14 | 17 | 21 | 24 | 27 | 32 | 37 |
| **9500** | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 11 | 13 | 16 | 20 | 23 | 26 | 30 | 35 |
| **10000** | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 12 | 15 | 19 | 22 | 25 | 28 | 34 |

## TABLE LXXVI (continued)

### DEVIATION

**d Table**

| Distance $OT$ in Yards | $T$ in Mils 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 | 900 | 1000 | 1100 | 1200 | 1300 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1000** | 10 | 20 | 30 | 39 | 48 | 57 | 65 | 72 | 79 | 85 | 90 | 94 | 97 |
| **1100** | 9 | 18 | 27 | 35 | 44 | 51 | 59 | 65 | 72 | 77 | 82 | 86 | 89 |
| **1200** | 8 | 17 | 25 | 32 | 40 | 47 | 54 | 60 | 66 | 71 | 75 | 78 | 81 |
| **1300** | 8 | 15 | 23 | 30 | 37 | 44 | 50 | 55 | 61 | 65 | 69 | 72 | 75 |
| **1400** | 7 | 14 | 21 | 28 | 34 | 40 | 46 | 51 | 56 | 60 | 64 | 67 | 70 |
| **1500** | 7 | 13 | 20 | 26 | 32 | 38 | 43 | 48 | 52 | 56 | 60 | 63 | 65 |
| **1600** | 6 | 12 | 18 | 24 | 30 | 35 | 40 | 45 | 49 | 53 | 56 | 59 | 61 |
| **1700** | 6 | 12 | 17 | 23 | 28 | 33 | 38 | 42 | 46 | 50 | 53 | 55 | 57 |
| **1800** | 6 | 11 | 16 | 22 | 27 | 31 | 36 | 40 | 44 | 47 | 50 | 52 | 54 |
| **1900** | 5 | 10 | 16 | 21 | 25 | 30 | 34 | 38 | 41 | 45 | 47 | 50 | 51 |
| **2000** | 5 | 10 | 15 | 20 | 24 | 23 | 32 | 36 | 39 | 42 | 45 | 47 | 49 |
| **2100** | 5 | 9 | 14 | 19 | 23 | 27 | 31 | 34 | 37 | 40 | 43 | 45 | 46 |
| **2200** | 5 | 9 | 13 | 18 | 22 | 26 | 29 | 33 | 36 | 38 | 41 | 43 | 44 |
| **2300** | 4 | 9 | 13 | 17 | 21 | 25 | 28 | 31 | 34 | 37 | 39 | 41 | 42 |
| **2400** | 4 | 8 | 12 | 16 | 20 | 24 | 27 | 30 | 33 | 35 | 37 | 39 | 41 |
| **2500** | 4 | 8 | 12 | 16 | 19 | 23 | 26 | 29 | 31 | 34 | 36 | 38 | 39 |
| **2600** | 4 | 8 | 11 | 15 | 18 | 22 | 25 | 28 | 30 | 33 | 35 | 36 | 37 |
| **2700** | 4 | 7 | 11 | 14 | 18 | 21 | 24 | 27 | 29 | 31 | 33 | 35 | 36 |
| **2800** | 4 | 7 | 11 | 14 | 17 | 20 | 23 | 26 | 28 | 30 | 32 | 34 | 35 |
| **2900** | 3 | 7 | 10 | 13 | 17 | 20 | 22 | 25 | 27 | 29 | 31 | 32 | 34 |
| **3000** | 3 | 7 | 10 | 13 | 16 | 19 | 22 | 24 | 26 | 28 | 30 | 31 | 32 |
| **3200** | 3 | 6 | 9 | 12 | 15 | 18 | 20 | 23 | 25 | 26 | 28 | 29 | 30 |
| **3400** | 3 | 6 | 9 | 11 | 14 | 17 | 19 | 21 | 23 | 25 | 26 | 28 | 29 |
| **3600** | 3 | 6 | 8 | 11 | 13 | 16 | 18 | 20 | 22 | 24 | 25 | 26 | 27 |
| **3800** | 3 | 5 | 8 | 10 | 13 | 15 | 17 | 19 | 21 | 22 | 24 | 25 | 26 |
| **4000** | 2 | 5 | 7 | 10 | 12 | 14 | 16 | 18 | 20 | 21 | 22 | 24 | 24 |
| **4500** | 2 | 4 | 7 | 9 | 11 | 13 | 14 | 16 | 18 | 19 | 20 | 21 | 22 |
| **5000** | 2 | 4 | 6 | 8 | 10 | 11 | 13 | 14 | 16 | 17 | 18 | 19 | 19 |
| **5500** | 2 | 4 | 5 | 7 | 9 | 10 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| **6000** | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 13 | 14 | 15 | 16 | 16 |
| **6500** | 2 | 3 | 5 | 6 | 7 | 9 | 10 | 11 | 12 | 13 | 14 | 14 | 15 |
| **7000** | 1 | 3 | 4 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 13 | 14 |
| **7500** | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 | 11 | 12 | 13 | 13 |
| **8000** | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 11 | 12 | 12 |

(Source: *Abbreviated Firing Tables*, U. S. War Department Technical Manual, No. 6-215. Page 14.)

## TABLE LXXVII

### *NATURAL FUNCTONS OF ANGLES IN MILS*

| Read Down | sin | cos | tan | cot | Read Up | Read Down | sin | cos | tan | cot | Read Up |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **0** | .0000 | 1.0000 | .0000 | | **1600** | **400** | .3827 | .9239 | .4142 | 2.414 | **1200** |
| **10** | .0098 | 1.0000 | .0098 | 101.9 | **90** | **10** | .3917 | .9201 | .4258 | 2.349 | **90** |
| **20** | .0196 | .9998 | .0196 | 50.92 | **80** | **20** | .4008 | .9162 | .4374 | 2.286 | **80** |
| **30** | .0295 | .9996 | .0295 | 33.94 | **70** | **30** | .4097 | .9122 | .4492 | 2.226 | **70** |
| **40** | .0393 | .9992 | .0393 | 25.45 | **60** | **40** | .4187 | .9081 | .4610 | 2.169 | **60** |
| **50** | .0491 | .9988 | .0491 | 20.36 | **50** | **50** | .4276 | .9040 | .4730 | 2.114 | **50** |
| **60** | .0589 | .9983 | .0590 | 16.96 | **40** | **60** | .4364 | .8998 | .4850 | 2.062 | **40** |
| **70** | .0687 | .9976 | .0688 | 14.53 | **30** | **70** | .4452 | .8954 | .4972 | 2.011 | **30** |
| **80** | .0785 | .9969 | .0787 | 12.71 | **20** | **80** | .4540 | .8910 | .5095 | 1.963 | **20** |
| **90** | .0882 | .9961 | .0886 | 11.29 | **10** | **90** | .4627 | .8865 | .5220 | 1.916 | **10** |
| **100** | .0980 | .9952 | .0985 | 10.15 | **1500** | **500** | .4714 | .8819 | .5345 | 1.871 | **1100** |
| **10** | .1078 | .9942 | .1084 | 9.224 | **90** | **10** | .4800 | .8773 | .5472 | 1.827 | **90** |
| **20** | .1175 | .9931 | .1184 | 8.449 | **80** | **20** | .4886 | .8725 | .5600 | 1.786 | **80** |
| **30** | .1273 | .9919 | .1283 | 7.793 | **70** | **30** | .4972 | .8677 | .5730 | 1.745 | **70** |
| **40** | .1370 | .9906 | .1383 | 7.230 | **60** | **40** | .5057 | .8627 | .5861 | 1.706 | **60** |
| **50** | .1467 | .9892 | .1483 | 6.741 | **50** | **50** | .5141 | .8577 | .5994 | 1.668 | **50** |
| **60** | .1564 | .9877 | .1584 | 6.314 | **40** | **60** | .5225 | .8526 | .6128 | 1.632 | **40** |
| **70** | .1661 | .9861 | .1685 | 5.936 | **30** | **70** | .5308 | .8475 | .6264 | 1.596 | **30** |
| **80** | .1758 | .9844 | .1786 | 5.600 | **20** | **80** | .5391 | .8422 | .6401 | 1.562 | **20** |
| **90** | .1855 | .9827 | .1887 | 5.299 | **10** | **90** | .5474 | .8369 | .6541 | 1.529 | **10** |
| **200** | .1951 | .9808 | .1989 | 5.027 | **1400** | **600** | .5556 | .8315 | .6682 | 1.497 | **1000** |
| **10** | .2047 | .9788 | .2091 | 4.781 | **90** | **10** | .5637 | .8260 | .6825 | 1.465 | **90** |
| **20** | .2143 | .9768 | .2194 | 4.558 | **80** | **20** | .5718 | .8204 | .6970 | 1.435 | **80** |
| **30** | .2239 | .9746 | .2297 | 4.353 | **70** | **30** | .5798 | .8148 | .7117 | 1.405 | **70** |
| **40** | .2335 | .9724 | .2401 | 4.165 | **60** | **40** | .5878 | .8090 | .7265 | 1.376 | **60** |
| **50** | .2430 | .9700 | .2505 | 3.992 | **50** | **50** | .5957 | .8032 | .7417 | 1.348 | **50** |
| **60** | .2525 | .9676 | .2610 | 3.832 | **40** | **60** | .6036 | .7973 | .7570 | 1.321 | **40** |
| **70** | .2620 | .9651 | .2715 | 3.684 | **30** | **70** | .6114 | .7914 | .7725 | 1.294 | **30** |
| **80** | .2714 | .9625 | .2820 | 3.546 | **20** | **80** | .6191 | .7853 | .7883 | 1.268 | **20** |
| **90** | .2809 | .9597 | .2927 | 3.417 | **10** | **90** | .6268 | .7792 | .8044 | 1.243 | **10** |
| **300** | .2903 | .9569 | .3034 | 3.297 | **1300** | **700** | .6344 | .7730 | .8207 | 1.219 | **900** |
| **10** | .2997 | .9540 | .3141 | 3.184 | **90** | **10** | .6420 | .7667 | .8372 | 1.194 | **90** |
| **20** | .3090 | .9511 | .3249 | 3.078 | **80** | **20** | .6495 | .7604 | .8541 | 1.171 | **80** |
| **30** | .3183 | .9480 | .3358 | 2.978 | **70** | **30** | .6569 | .7540 | .8712 | 1.148 | **70** |
| **40** | .3276 | .9448 | .3468 | 1.884 | **60** | **40** | .6643 | .7475 | .8886 | 1.125 | **60** |
| **50** | .3369 | .9415 | .3578 | 2.795 | **50** | **50** | .6716 | .7410 | .9064 | 1.103 | **50** |
| **60** | .3461 | .9382 | .3689 | 2.711 | **40** | **60** | .6788 | .7343 | .9244 | 1.082 | **40** |
| **70** | .3553 | .9348 | .3801 | 2.631 | **30** | **70** | .6860 | .7276 | .9428 | 1.061 | **30** |
| **80** | .3645 | .9312 | .3914 | 2.555 | **20** | **80** | .6931 | .7209 | .9615 | 1.040 | **20** |
| **90** | .3736 | .9276 | .4028 | 2.483 | **10** | **90** | .7001 | .7140 | .9806 | 1.020 | **10** |
| **400** | .3827 | .9239 | .4142 | 2.414 | **1200** | **800** | .7071 | .7071 | 1.0000 | 1.000 | **800** |
| | **cos** | **sin** | **cot** | **tan** | | | **cos** | **sin** | **cot** | **tan** | |